52-nd Polish Mathematical Olympiad 2000/01

Second Round

February 23–24, 2001

First Day

1. Let $k, n > 1$ be integers such that the number $p = 2k - 1$ is prime. Prove that, if the number $\binom{n}{2} - \binom{k}{2}$ is divisible by p, then it is divisible by p^2.

2. Points A, B, C with $AB < BC$ lie in this order on a line. Let $ABDE$ be a square. The circle with diameter AC intersects the line DE at points P and Q with P between D and E. The lines AQ and BD intersect at R. Prove that $DP = DR$.

3. Let $n \geq 3$ be a positive integer. Prove that a polynomial of the form
$$x^n + a_{n-3}x^{n-3} + a_{n-4}x^{n-4} + \cdots + a_1 x + a_0,$$
where at least one of the real coefficients $a_0, a_1, \ldots, a_{n-3}$ is nonzero, cannot have all real roots.

Second Day

4. Find all integers $n \geq 3$ for which the following statement is true:
Any arithmetic progression $a_1, \ldots, a_n$ with n terms for which $a_1 + 2a_2 + \cdots + na_n$ is rational contains at least one rational term.

5. In a triangle ABC, I is the incenter and D the intersection point of AI and BC. Show that $AI + CD = AC$ if and only if $\angle B = 60^\circ + \frac{1}{3}\angle C$.

6. For a positive integer n, let A_n and B_n be the families of n-element subsets of $S_n = \{1, 2, \ldots, 2n\}$ with respectively even and odd sums of elements. Compute $|A_n| - |B_n|$.

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