50-th Polish Mathematical Olympiad 1998/99

First Round

September – December, 1998

1. Prove that among the numbers $50^n + (50n+1)^{50}$, where $n \in \mathbb{N}$, there are infinitely many composite numbers.

2. If a, b, c, d are real numbers, prove the inequality

$$(a+b+c+d)^2 \leq 3(a^2+b^2+c^2+d^2) + 6ab.$$

3. Let ABC be an isosceles triangle with $\angle A = 90^\circ$. Point D is taken on side BC such that $BD = 2CD$, and E is the projection of B onto the line AD. Compute $\angle CED$.

4. Suppose that x, y are real numbers such that $x+y$, x^2+y^2, x^3+y^3, and x^4+y^4 are integers. Prove that x^n+y^n is an integer for all $n \in \mathbb{N}$.

5. Determine all positive integers x, y satisfying $y^x = x^{50}$.

6. Diagonals AC and BD of a convex quadrilateral $ABCD$ meet at P. Let M be the midpoint of AB and let the line MP meet CD at Q. Prove that the ratio of the areas of the triangles BCP and ADP equals $CQ : DQ$.

7. Let $n \geq 2$ be an integer. Find all polynomials $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ having exactly n real roots not exceeding -1 and satisfying

$$a_0^2 + a_1 a_n = a_n^2 + a_0 a_{n-1}.$$

8. Let S be a set of $n \geq 2$ elements. Find the smallest k for which there exist subsets $A_1, A_2, \dots, A_k$ of S with the following property: For any two elements a, b of S there exists $j \in \{1, 2, \dots, k\}$ such that A_j contains exactly one of the elements a, b.

9. Suppose that D, E, F are points on the sides BC, CA, AB of a triangle ABC respectively such that the incircles of the triangles AEF, BFD, CDE are tangent to the incircle of the triangle DEF. Show that the lines AD, BE, CF are concurrent.

10. Given $x_1 > 0$, the sequence (x_n) is defined by

$$x_{n+1} = x_n + \frac{1}{x_n^2} \quad \text{for } n \geq 1.$$

Prove that the limit $\lim_{n \to \infty} \frac{x_n}{\sqrt[3]{n}}$ exists and find it.

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11. There are a white ball and a black ball in an urn. In addition, we are given 50 white and 50 black balls. We repeat the following procedure 50 times: we choose at random one ball from the urn and return it to the urn together with another ball of the same color. What will be the most probable numbers of white balls in the urn?

12. All vertices of a cube of edge a lie on the surface of a regular tetrahedron of edge 1. Find all possible values of a.

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