49-th Polish Mathematical Olympiad 1997/98

First Round

September 11 – December 10, 1997

1. Solve the system of equations:

$$\begin{aligned} |x-y| - \frac{|x|}{x} &= -1, \\ |2x-y| + |x+y-1| + |x-y| + y - 1 &= 0. \end{aligned}$$

2. Let H be the orthocenter of a triangle inscribed in a circle with center O. Given that $AO = AH$, find the measure of $\angle CAB$.

3. The sequences (a_n), (b_n), (c_n) are given by $a_1 = 4$ and for $n \geq 1$,

$$a_{n+1} = a_n(a_n - 1), \quad 2^{b_n} = a_n, \quad 2^{n-c_n} = b_n.$$

Prove that the sequence (c_n) is bounded.

4. Let a be a positive number. Determine all real numbers c with the property that, for any positive numbers x, y, the following inequality holds:

$$(c-1)x^{a+1} \leq (cy - x)y^a.$$

5. Solve the equation $|\tan^n x - \cot^n x| = 2n|\cot 2x|$, where n is a given positive integer.

6. In a triangle ABC with $AB > AC$, D is the midpoint of BC and E is an arbitrary point on side AC. Points P and Q are the orthogonal projections of B and E onto AD, respectively. Show that $BE = AE + AC$ if and only if $AD = PQ$.

7. Let m, n be given positive integers and $A = \{1, 2, \dots, n\}$. Determine the number of functions $f : A \to A$ attaining exactly m values such that

$$f(f(k)) = f(k) \leq f(l) \quad \text{for all } k, l \in A \text{ with } k \leq l.$$

8. Determine if there exists a convex polyhedron having exactly k edges and a plane not passing through any vertex and cutting r edges such that $3r > 2k$.

9. Define $a_0 = 0.91$ and $a_k = 0.\underbrace{99\dots9}_{2^k}\underbrace{00\dots0}_{2^k-1}1$ for $k > 0$. Compute $\lim_{n\to\infty} a_0 a_1 \cdots a_n$.

10. The medians AD, BE, CF of a triangle ABC meet at G. Prove that if the quadrilaterals $AFGE$ and $BDGF$ are cyclic, then the triangle ABC is equilateral.

11. In a tennis tournament n players took part. Any two players played a match (no draws). Prove that there is a player A such that for any other player B, A either defeated B or there is a player C who defeated B but lost to A.

12. Let $g(k)$ denote the greatest prime divisor of an integer k if $|k| \geq 2$, and $g(-1) = g(0) = g(1) = 1$. Find if there exists a non-constant polynomial W with integer coefficients such that the set $\{g(W(x)) \mid x \in \mathbb{Z}\}$ is finite.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com