48-th Polish Mathematical Olympiad 1996/97

First Round

September – December 1996

1. Solve the system of equations $x|x|+y|y| = [x]+[y] = 1$.

2. Let P be a point inside a parallelogram $ABCD$ such that $\angle ABP = \angle ADP$. Prove that $\angle PAB = \angle PCB$.

3. Let $a, b \geq 1$, $c \geq 0$ be real numbers and $n \geq 1$ be an integer. Prove that
$$(ab+c)^n - c \leq a^n\left((b+c)^n - c\right).$$

4. Prove that an integer $n \geq 2$ is composite if and only if there are positive integers a, b, x, y with $a+b = n$ and $\frac{x}{a} + \frac{y}{b} = 1$.

5. The angle bisectors of the angles A, B, C of a triangle ABC meet the opposite sides at D, E, F and the circumcircle of $\triangle ABC$ at K, L, M, respectively. Prove that
$$\frac{AD}{DK} + \frac{BE}{EL} + \frac{CF}{FM} \geq 9.$$

6. If $P(x)$ is a polynomial of degree n such that $P(k) = 1/k$ for $k = 1, 2, 4, 8, \dots, 2^n$, determine $P(0)$.

7. Find the supremum of volumes of tetrahedra contained in a ball of a given radius, whose one edge is a diameter of the ball.

8. Let a_n denote the number of all nonempty subsets of $\{1, 2, \dots, 6n\}$, whose sum of elements gives the remainder 5 when divided by 6. Also, let b_n be the number of all nonempty subsets of $\{1, 2, \dots, 7n\}$ whose product of elements gives the remainder 5 when divided by 7. Find a_n/b_n.

9. Find all functions $f : [1, \infty) \to [1, \infty)$ which satisfy:

   (i) $f(x+1) = \dfrac{f(x)^2 - 1}{x}$ for all $x \geq 1$;

   (ii) the function $g(x) = f(x)/x$ is bounded.

10. Let P, Q be points inside an acute-angled triangle ABC such that $\angle ACP = \angle BCQ$ and $\angle CAP = \angle BAQ$. Let D, E, F be the feet of the perpendiculars from P to BC, CA, AB, respectively. Prove that $\angle DEF = 90^\circ$ if and only if Q is the orthocenter of $\triangle BDF$.

11. Let m be a positive integer and $P(x)$ a non-constant polynomial with integer coefficients. Prove that if $P(x)$ has at least three distinct integer roots, then $P(x) + 5^m$ has at most one integer root.

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12. A group of n people noticed that, for some period of time, three of them might be going for a dinner together, each pair meeting at exactly one dinner. Prove that $n \equiv 1$ or $n \equiv 3 \pmod 6$.

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