47-th Polish Mathematical Olympiad 1995/96

First Round

September – December 1995

1. Find all positive integers n for which the equation $\tan x + \cot x = 2\sin nx$ has a real solution.

2. A natural number is *palindromic* if it is equal to the number obtained by reading its decimal representation from right to left. Let $x_1, x_2, x_3, \dots$ be the increasing sequence of all palindromic numbers. Find all primes which divide at least one of the differences $x_{n+1} - x_n$.

3. In a group of kn persons $(k, n \in \mathbb{N})$, everybody knows more than $(k-1)n$ of the others. Prove that there is a group of $k+1$ persons which all know each other.

4. A line tangent to the incircle of an equilateral triangle ABC intersects AB and AC at D and E respectively. Prove that $\frac{AD}{DB} + \frac{AE}{EC} = 1$.

5. In a triangle ABC, the angle $\angle CAB = \alpha$ is obtuse. Let PQ be any segment whose midpoint is A. Show that $(BP + CQ)\tan\frac{\alpha}{2} \geq BC$.

6. Two increasing sequences of positive integers are given: an arithmetic progression with difference $r > 0$ and a geometric progression with ratio $q > 1$, where q and r are coprime. Prove that if these two sequences have a common term, then they have infinitely many common terms.

7. Let a, b, c and p, q, r be nonnegative numbers satisfying $a+b+c = p+q+r = 1$ and $p, q, r \leq 1/2$. Prove that
$$8abc < pa + qb + rc$$
and find when equality occurs.

8. A ray of light starts from the center of a square and reflects from its sides (the angles of reflection and incidence are always equal). The ray never reaches a vertex of the square and, after some time, it returns to the center for the first time. Prove that the ray reflected from the sides of the square an odd number of times.

9. A polynomial with integer coefficients, when divided by $x^2 - 12x + 11$, gives the remainder $990x - 889$. Prove that this polynomial has no integer roots.

10. Prove that the equation $x^x = y^3 + z^3$ has infinitely many solutions in positive integers x, y, z.

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11. In a skiing jump competition 65 contestants jump in a previosly established order, each of them exactly once. We assume that their results are all different and that all possible final rankings are equaly likely. In each moment of the competition we call a person with the best score (at that moment) a leader. Let p be the probability that during the whole competition there was exactly one change of a leader. Prove that $p > 1/16$.

12. Find if there exist two congruent cubes with a common center such that each face of one cube and each face of the other cube have a common point.

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