46-th Polish Mathematical Olympiad 1994/95

First Round

September – December 1994

1. Find all pairst (x,y) of natural numbers such that the numbers $\dfrac{x+1}{y}$ and $\dfrac{y+1}{x}$ are natural.

2. For a positive integer $n \geq 2$, solve the following system of equations:

$$\begin{aligned} x_1|x_1| &= x_2|x_2| + (x_1-1)|x_1-1|, \\ x_2|x_2| &= x_3|x_3| + (x_2-1)|x_2-1|, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x_n|x_n| &= x_1|x_1| + (x_n-1)|x_n-1|. \end{aligned}$$

3. A quadrilateral with sides a,b,c,d is inscribed in a circle of radius R. Prove that if $a^2+b^2+c^2+d^2 = 8R^2$, then either one of the angles of the quadrilateral is right or its diagonals are perpendicular.

4. In some school 64 students participate in five different olympiads. In each olympiad at least 19 students take part, but none of the students participates in more than three olympiads. Prove that if every three olympiads have a common participant, then there are two olympiads having at least five common participants.

5. Prove that the following two conditions on positive numbers a,b are equivalent:

   (i) $\sqrt{a}+1 > \sqrt{b}$;

   (ii) for every $x > 1$, $ax + \dfrac{x}{x-1} > b$.

6. Let P be a point inside a triangle ABC. The rays AP, BP, CP intersect BC, CA, AB at A', B', C', respectively. Set $u = AP/PA'$, $v = BP/PB'$, $w = CP/PC'$. Express the product uvw in terms of the sum $u+v+w$.

7. (a) Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$, not identically equal to 0, such that $2f(f(x)) = f(x) \geq 0$ for all real x?

   (b) Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$, not identically equal to 0, such that $-1 \leq 2f(f(x)) = f(x) \leq 1$ for all real x?

8. In a regular pyramid with a regular n-gon as a base, a lateral face and the base form an angle α, while a lateral edge and the base form an angle β. Prove that

$$\sin^2\alpha - \sin^2\beta \leq \tan^2\frac{\pi}{2n}.$$

9. Let a and b be positive real numbers with the sum 1. If a^3 and b^3 are rational, show that so are a and b.

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10. Three distinct points are given on a line k. From each of these points we draw a pair of rays so that all the rays are on the same side of k. Every two of these three pairs of rays form a quadrilateral. Prove that if two of these quadrilaterals are tangent, then so is the third.

11. Let $n > m > 1$ be rational numbers. We randomly draw m distinct numbers from the set $\{1, 2, \dots, n\}$. Find the expected value of the difference between the largest and the smalles of the drawn numbers.

12. The sequence (x_n) is given by

$$x_1 = \frac{1}{2}, \qquad x_n = \frac{2n-3}{2n} x_{n-1} \text{ for all } n \geq 2.$$

Prove that for all $n \in \mathbb{N}$ it holds that $x_1 + x_2 + \cdots + x_n < 1$.

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