45-th Polish Mathematical Olympiad 1993/94

First Round

September – December 1993

1. Prove that there are no integers a,b,c,d, not all equal to 0, such that $a^2-b=c^2$ and $b^2-a=d^2$.

2. The sequence of functions $f_n:\mathbb{R}\to\mathbb{R}$ is given by $f_0(x)=|x|$ and, for each n,
$$f_{n+1}(x)=|f_n(x)-2| \qquad \text{for all } x.$$
Solve the equation $f_n(x)=1$, where n is a given positive integer.

3. Prove that if a,b,c are sides of a triangle, then
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\le\frac{1}{a+b-c}+\frac{1}{b+c-a}+\frac{1}{c+a-b}.$$

4. Let be given a point A inside a circle with center O, and a chord PQ through A which is not a diameter. Let p,q be the tangents to the circle at P,Q, respectively. The line l through A perpendicular to OA intersects p and q at K and L respectively. Prove that $AK=AL$.

5. Prove that if the polynomial x^3+ax^2+bx+c has three distinct real roots, then so does the polynomial
$$x^3+ax^2+\frac{1}{4}(a^2+b)x+\frac{1}{8}(ab-c).$$

6. Suppose that $f:\mathbb{R}\to\mathbb{R}$ is a continuous function such that for every real x there exists $n\in\mathbb{N}$ such that
$$\underbrace{f\circ f\circ\cdots\circ f}_{n}(x)=1.$$
Show that $f(1)=1$.

7. Outside a convex quadrilateral $ABCD$, similar triangles APB,BQC,CRD,DSA are constructed so that
$$\angle PAB=\angle QBC=\angle RCD=\angle SDA,$$
$$\angle PBA=\angle QCB=\angle RDC=\angle SAD.$$
Prove that if $ABCD$ is a parallelogram, then so is $PQRS$.

8. Let a,b,c be positive integers such that $b\mid a^3$, $c\mid b^3$ and $a\mid c^3$. Prove that $abc\mid(a+b+c)^{13}$.

9. There are $2n$ participants in a conference. Each of the persons is acquainted to at least n other persons. Prove that it is possible to accomodate the participants in n double rooms so that each of them would share a room with his/her acquaintance.

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10. Let p,q be nonnegative real numbers with $p+q=1$, and let m,n be positive integers. Prove that
$$(1-p^m)^n+(1-q^n)^m \geq 1.$$

11. Let R and r be the circumradius and inradius of a triangle of perimeter $2p$, respectively. Show that $p<2(R+r)$.

12. Prove that the sums of the opposite dihedral angles of a tetrahedron are equal if and only if the sums of the opposite edges of the tetrahedron are equal.

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