44-th Polish Mathematical Olympiad 1992/93

First Round

September – December 1992

1. Solve the following equation in real numbers:
$$\frac{(x^2-1)(|x|+1)}{x+\operatorname{sgn} x}=[x+1].$$

2. Let $n \geq 3$ be integer. Solve the system of equations:
$$\begin{aligned}
\tan x_1 + 3\cot x_1 &= 2\tan x_2,\\
\tan x_2 + 3\cot x_2 &= 2\tan x_3,\\
\dots\dots\dots\dots\; &\dots\; \dots\\
\tan x_n + 3\cot x_n &= 2\tan x_1.
\end{aligned}$$

3. Let $ABCDEF$ be a centrally symmetric hexagon. The lines AB and EF meet at A', the lines BC and AF meet at B', and the lines AB and CD meet at C'. Prove that
$$AB \cdot BC \cdot CD = AA' \cdot BB' \cdot CC'.$$

4. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y,
$$f(x+y) - f(x-y) = f(x)f(y).$$

5. Let A and C be distinct points in the plane. For every point B one constructs squares $ABKL$ and $BCMN$ outside the triangle ABC. Prove that the lines LM pass through a fixed point as B varies in the same halfplane determined by AC.

6. The sequence (x_n) is defined by $x_0 = 1992$ and
$$x_n = -\frac{1992}{n}\sum_{k=0}^{n-1} x_k$$
for each $n \geq 1$. Calculate $\displaystyle\sum_{n=0}^{1992} 2^n x_n$.

7. Consider the points $A_0(0,0,0)$, $A_1(1,0,0)$, $A_2(0,1,0)$ and $A_3(0,0,1)$ in space. Let the point P_{ij} $(i, j = 0,1,2,3)$ be defined by $\overrightarrow{A_0P_{ij}} = \overrightarrow{A_iA_j}$. Find the volume of the convex hull of points P_{ij}.

8. Given a positive integer n, determine the maximum possible value of the sum of natural numbers $k_1, k_2, \dots, k_n$ satisfying
$$k_1^3 + k_2^3 + \dots + k_n^3 \leq 7n.$$

9. Let a,b,c be reaql numbers. Prove the inequality

$$\begin{aligned} & (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2) \quad \leq \\ \leq \quad & (a+b-c)^2(b+c-a)^2(c+a-b)^2. \end{aligned}$$

10. Let $\mathscr{C}$ be a cube and let $f:\mathscr{C}\to\mathscr{C}$ be a surjection such that $|PQ|\geq|f(P)f(Q)|$ for all $P,Q\in\mathscr{C}$. Prove that f is an isometry.

11. Six pawns are randomly placed on an $n\times n$ chessboard. Let p_n be the probability that at least two of the pawns lie in the same row or column. Find $\lim\limits_{n\to\infty} np_n$.

12. Prove that the polynomial x^n+4 is expressible as the product of two non-constant polynomials with integer coefficients if and only if $4\mid n$.

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