57-th Polish Mathematical Olympiad 2005/06

First Round

September 12 – December 5, 2005

1. Determine all nonnegative integers n for which $2^n + 105$ is a perfect square.

2. Solve the equation $\sqrt[5]{x} = \left[\sqrt[5]{3x}\right]$ in nonnegative real numbers.

3. An acute-angled triangle ABC is inscribed in a circle with center O. Point D is the projection of C onto AB, and points E and F are the projections of the point D onto AC and BC, respectively. Prove that the area of quadrilateral $EOFC$ equals half the area of triangle ABC.

4. The participants of a mathematical ocmpetition were solving six problems. Each problem was marked with 6, 5, 2 or 0 points. It turned out that for every two participants A and B there are two problems, such that on each of them A and B obtained different scores. Find the largest possible number of participants for which this is possible.

5. Let a, b be real numbers. Consider the functions
$$f(x) = ax + b|x| \quad \text{and} \quad g(x) = ax - b|x|.$$
Prove that if $f(f(x)) = x$ for every $x \in \mathbb{R}$, then $g(g(x)) = x$ for every $x \in \mathbb{R}$.

6. A line passes through the orthocenter H of an acute-angled triangle ABC and meets the sides AC and BC at D and E, respectively. The line through H perpendicular to DE intersects the line AB at point F. Prove that $\dfrac{DH}{HE} = \dfrac{AF}{FB}$.

7. A prime number $p > 3$ and positive integers a, b, c satisfy $a + b + c = p + 1$ and the number $a^3 + b^3 + c^3 - 1$ is divisible by p. Show that at least one of the numbers a, b, c is equal to 1.

8. A tetrahedron $ABCD$ is circumscribed to a sphere with center S and radius 1 such that $SA \geq SB \geq SC$. Show that $SA > \sqrt{5}$.

9. Let $k_1 < k_2 < \cdots < k_m$ be nonnegative integers. Define $n = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_m}$. Find the number of odd coefficients of the polynomial $P(x) = (x+1)^n$.

10. Positive numbers a, b, c satisfy the equality $ab + bc + ca = 3$. Prove that
$$a^3 + b^3 + c^3 + 6abc \geq 9.$$

11. In a concave quadrilateral $ABCD$ the interior angle at A is greater than 180° and $AB \cdot CD = AD \cdot BC$. Point P is symmetric to A with respect to BD. Prove that $\angle PCB = \angle ACD$.

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12. For a given positive integer a_0 define the sequence (a_n) by

$$a_{i+1} = \begin{cases} a_i/2 & \text{if } a_i \text{ is even,} \\ 3a_i - 1 & \text{if } a_i \text{ is odd,} \end{cases} \quad , i = 0, 1, 2, \dots$$

Prove that if n is a natural number such that $a_n = a_0$, then $2^n > a_0$.

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