55-th Polish Mathematical Olympiad 2003/04

First Round

September – December, 2003

1. Let be given a polygon with rational side lengths and all angles equal to 90° or 270°. A ray of light starts at one of the vertices of the polygon and goes in the direction of the interior angle bisector at that vertex. The ray reflects according to the law of reflection. Prove that the ray will eventually enter some vertex of the polygon.

2. Decide whether there exist a prime p and nonnegative integers x,y,z such that $(12x+5)(12y+7)=p^z$.

3. Find all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that for all rational x,y
$$f(x^2+y)=xf(x)+f(y).$$

4. An acute-angled triangle ABC is given. Consider all equilateral triangles XYZ in this plane such that the points A,B,C lie on the segments YZ,ZX,XY, respectively. Prove that the centers of triangles XYZ lie on a single circle.

5. For positive integers m and n, let $N(m,n)$ denote the number of non-decreasing sequences of m terms from the set $\{1,2,\dots,n\}$. Show that $N(m,n+1)=N(n,m+1)$.

6. Suppose c is a real number such that the polynomial $P(x)=x^5-5x^3+4x-c$ has five real zeroes x_1,x_2,x_3,x_4,x_5. Compute in terms of c the sum of the absolute values of the coefficients of the polynomial
$$Q(x)=(x-x_1^2)(x-x_2^2)(x-x_3^2)(x-x_4^2)(x-x_5^2).$$

7. Find all positive integer solutions of the equation $a^2+b^2=c^2$ such that a and c are prime and b is a product of at most four prime numbers.

8. Point P lies inside a tetrahedron $ABCD$. Prove that
$$\angle APB+\angle BPC+\angle CPD+\angle DPA>360^\circ.$$

9. Let be given nonconstant polynomials $W_1(x),W_2(x),\dots,W_n(x)$ with integer coefficients. Prove that for some integer a all the numbers $W_1(a),W_2(a),\dots,W_n(a)$ are composite.

10. A convex polygon has an even number of sides. The length of each side is either 2 or 3, and the number of sides of each length is even. Show that there exist two vertices of the polygon that bisect its perimeter.

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11. Let O be the circumcenter of an isosceles trapezoid $ABCD$ with the bases AB and CD. Points K,L,M,N respectively lie on the sides AB,BC,CD,DA so that $KLMN$ is a rhombus. Prove that O lies on the line KM.

12. Let $n \geq 5$ be an integer. Find the number of solutions in real numbers $x_1,\dots,x_n$ of the system

$$x_{i-2}^3 + x_{i-1}^3 + x_i^3 = x_i^4 + x_{i+1}^3 + x_{i+2}^2 \quad \text{for } i = 1,2,\dots,n,$$

where $x_{-1} = x_{n-1}$, $x_0 = x_n$, $x_1 = x_{n+1}$, $x_2 = x_{n+2}$.

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