10-th Pan-African Mathematical Olympiad

Cape Town, South Africa, 2000

First Day

1. Solve the equation $\sin^3 x(1+\cot x)+\cos^3 x(1+\tan x)=\cos 2x$.

2. Define the polynomials $P_0, P_1, P_2, \dots$ by

$$P_0(x)=x^3+213x^2-67x-2000,$$
$$P_n(x)=P_{n-1}(x-n), \quad x\in\mathbb{R}.$$

   Find the coefficient at x in $P_{21}(x)$.

3. Given that $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{1334}+\frac{1}{1335}=\frac{p}{q}$, where p and q are coprime natural numbers, prove that p is divisible by 2003.

Second Day

4. Real numbers a,b,c satisfy $a^2+b^2=c^2$. Find all real numbers x,y,z such that

$$x^2+y^2=z^2 \quad \text{and} \quad (x+a)^2+(y+b)^2=(z+c)^2.$$

5. Let P be a point outside a circle γ and let PA and PB be the tangents to γ (where $A,B\in\gamma$). A line through P intersects γ at Q and R, and S is a point on γ such that $BS \parallel QR$. Prove that SA bisects QR.

6. A company has five directors. The regulations of the company require that any majority (three or more) of the directors should be able to open the strongroom, but any minority should not be able to do so. The strongroom is equipped with ten locks and can only be opened when keys to all ten locks are available. Find all positive integers n such that it is possible to give each director a set of keys to n different locks according to the regulations of the company.

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