The Niels Henrik Abel Contest 1999

Final Round – March 12, 1999

1. (a) Find a function f such that $f(t^2+t+1)=t$ for all real $t \geq 0$.

   (b) If a,b,c,d,e are real numbers, prove the inequality

$$a^2+b^2+c^2+d^2 \geq a(b+c+d+e).$$

2. (a) Find all integers m and n such that $2m^2+n^2=2mn+3n$.

   (b) If a,b,c are positive integers such that $b \mid a^3$, $c \mid b^3$ and $a \mid c^3$, prove that $abc \mid (a+b+c)^{13}$.

3. An isosceles triangle ABC with $AB=AC$ and $\angle A = 30^\circ$ is inscribed in a circle with center O. Point D lies on the shorter arc AC so that $\angle DOC = 30^\circ$, and point G lies on the shorter arc AB so that $DG=AC$ and $AG<BG$. The line BG intersects AC and AB at E and F, respectively.

   (a) Prove that triangle AFG is equilateral.

   (b) Find the ratio between the areas of triangles AFE and ABC.

4. For every nonempty subset R of $S=\{1,2,\dots,10\}$, we define the *alternating sum* $A(R)$ as follows: If $r_1,r_2,\dots,r_k$ are the elements of R in the increasing order, then $A(R)=r_k-r_{k-1}+r_{k-2}-\cdots+(-1)^{k-1}r_1$.

   (a) Is it possible to partition S into two sets having the same alternating sum?

   (b) Determine the sum $\sum_R A(R)$, where R runs over all nonempty subsets of S.

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