The Niels Henrik Abel Contest 1998

Final Round – March 12, 1998

1. Let $a_0, a_1, a_2, \dots$ be an infinite sequence of positive integers such that $a_0 = 1$ and $a_i^2 > a_{i-1}a_{i+1}$ for all $i > 0$.

   (a) Prove that $a_i < a_1^i$ for all $i > 1$.

   (b) Prove that $a_i > i$ for all i.

2. Let be given an $n \times n$ chessboard, $n \in \mathbb{N}$. We wish to tile it using particular tetraminos which can be rotated. For which n is this possible if we use

   (a) T-tetraminos

   (b) both kinds of L-tetraminos?

3. Let n be a positive integer.

   (a) Prove that $1^5 + 3^5 + 5^5 + \cdots + (2n-1)^5$ is divisible by n.

   (b) Prove that $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3$ is divisible by n^2.

4. Let A, B, P be points on a line l, with P outside the segment AB. Lines a and b pass through A and B and are perpendicular to l. A line m through P, which is neither parallel nor perpendicular to l, intersects a and b at Q and R, respectively. The perpendicular from B to AR meets a and AR at S and U, and the perpendicular from A to BQ meets b and BQ at T and V, respectively.

   (a) Prove that P, S, T are collinear.

   (b) Prove that P, U, V are collinear.

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