The Niels Henrik Abel Contest 1997

Final Round

1. We call a positive integer n *happy* if there exist integers a, b such that $a^2 + b^2 = n$. If t is happy, show that

   (a) $2t$ is happy;

   (b) $3t$ is not happy.

2. (a) Let P be an interior point of an equilateral triangle ABC, and let Q, R, S be the feet of perpendiculars from P to AB, BC, CA, respectively. Show that the sum $PQ + PR + PS$ is independent of the choice of P.

   (b) Let A, B, C be different points on a circle such that $AB = AC$. Point E lies on the segment BC, and $D \neq A$ is the point of intersection of the circle and line AE. Show that the product $AE \cdot AD$ is independent of the choice of E.

3. (a) Each subset of 97 out of 1997 given real numbers has positive sum. Show that the sum of all the 1997 numbers is positive.

   (b) Ninety-one students in a school are distributed in three classes. Each student took part in a competition. It is known that among any six students of the same sex some two got the same number of points. Show that here are four students of the same sex who are in the same class and who got the same number of points.

4. Let $p(x)$ be a polynomial with integer coefficients. Suppose that there exist different integers a and b such that $f(a) = b$ and $f(b) = a$. Show that the equation $f(x) = x$ has at most one integral solution.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com