The Niels Henrik Abel Contest 1995

Final Round

1. (a) Let a function f satisfy $f(1)=1$ and $f(1)+f(2)+\cdots+f(n)=n^2f(n)$ for all $n\in\mathbb{N}$. Determine $f(1995)$.

   (b) Prove that if $\left(x+\sqrt{x^2+1}\right)\left(y+\sqrt{y^2+1}\right)=1$ for real numbers x,y, then $x+y=0$.

2. (a) Two circles k_1,k_2 touch each other at P, and touch a line l at A and B respectively. Line AP meets k_2 at C. Prove that BC is perpendicular to l.

   (b) Two circles of the same radii intersect in two distinct points A and B. A line P, not touching any of the circles, intersects the circles again at A and B. Prove that Q lies on the perpendicular bisector of AB.

3. Show that there exists a sequence $x_1,x_2,\dots$ of natural numbers in which every natural number occurs exactly once, such that the sums $\sum_{i=1}^{n}\frac{1}{x_i}$, $n=1,2,3,\dots$, include all natural numbers.

4. Let x_i,y_i be positive real numbers, $i=1,2,\dots,n$. Prove that

$$\left(\sum_{i=1}^{n}(x_i+y_i)^2\right)\left(\sum_{i=1}^{n}\frac{1}{x_iy_i}\right)\geq 4n^2.$$

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