The Niels Henrik Abel Contest 1994

Final Round

1. (a) In a half-ball of radius 3 is inscribed a cylinder with base lying on the base plane of the half-ball, and another such cylinder with equal volume. If the base-radius of the first cylinder is $\sqrt{3}$, what is the base-radius of the other one?

   (b) Let C be a point on the prolongation of the diameter AB of a circle. A line through C is tangent to the circle at point N. The bisector of $\angle ACN$ meets the lines AN and BN at P and Q respectively. Prove that $PN = QN$.

2. (a) Find all primes p,q,r and natural numbers n such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{1}{n}$.

   (b) Find all integers x,y,z such that $x^3+5y^3=9z^3$.

3. (a) Let $x_1,x_2,\dots,x_{1994}$ be positive real numbers. Prove that

$$\left(\frac{x_1}{x_2}\right)^{\frac{x_1}{x_2}}\left(\frac{x_2}{x_3}\right)^{\frac{x_2}{x_3}}\cdots\left(\frac{x_{1993}}{x_{1994}}\right)^{\frac{x_{1993}}{x_{1994}}} \geq \left(\frac{x_1}{x_2}\right)^{\frac{x_2}{x_1}}\left(\frac{x_2}{x_3}\right)^{\frac{x_3}{x_2}}\cdots\left(\frac{x_{1993}}{x_{1994}}\right)^{\frac{x_{1994}}{x_{1993}}}.$$

   (b) Prove that there is no function $f:\mathbb{Z}\to\mathbb{Z}$ such that $f(f(x)) = x+1$ for all x.

4. (a) In a group of 20 people, each person sends a letter to 10 of the others. Prove that there are two persons who send a letter to each other.

   (b) Finitely many cities are connected by one-way roads. For any two cities it is possible to come from one of them to the other (with possible transfers), but not necessarily both ways. Prove that there is a city which can be reached from any other city, and that there is a city from which any other city can be reached.

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