The Niels Henrik Abel Contest 1993

Final Round

1. (a) Let $ABCD$ be a convex quadrilateral and A', B', C', D' be the midpoints of AB, BC, CD, DA, respectively. Let a, b, c, d denote the areas of quadrilaterals into which lines $A'C'$ and $B'D'$ divide the quadrilateral $ABCD$ (where a corresponds to vertex A etc.). Prove that $a + c = b + d$.

   (b) Given a triangle with sides of lengths a, b, c, prove that

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} < 2.$$

2. If a, b, c, d are real numbers with $b < c < d$, prove that

$$(a+b+c+d)^2 > 8(ac+bd).$$

3. The Fermat-numbers are defined by $F_n = 2^{2^n} + 1$ for $n \in \mathbb{N}$.

   (a) Prove that $F_n = F_{n-1}F_{n-2} \cdots F_1 F_0 + 2$ for $n > 0$.

   (b) Prove that any two different Fermat-numbers are coprime.

4. Each of the 8 vertices of a given cube is given a value 1 or -1. Each of the 6 faces is given the value of product of its four vertices. Let A be the sum of all the 14 values. Which are the possible values of A?

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