The Niels Henrik Abel Contest 2004

Final Round – March 11, 2004

1. (a) If m is a positive integer, prove that 2^m cannot be written as a sum of two or more consecutive natural numbers.

   (b) Let $a_1, a_2, a_3, \dots$ be a strictly increasing sequence of positive integers. A number a_n in the sequence is said to be *lucky* if it is the sum of several (not necessarily distinct) smaller terms of the sequence, and *unlucky* otherwise. (For example, in the sequence $4, 6, 14, 15, 25, \dots$ numbers $4, 6, 15$ are unlucky, while $14 = 4+4+6$ and $25 = 4+6+15$ are lucky.) Prove that there are only finitely many unlucky numbers in the sequence.

2. (a) Prove that $(x+y+z)^2 \le 3(x^2+y^2+z^2)$ for any real numbers x, y, z.

   (b) If positive numbers a, b, c satisfy $a+b+c \ge abc$, prove that $a^2+b^2+c^2 \ge \sqrt{3}abc$.

3. In a quadrilateral $ABCD$ with $\angle A = 60^\circ$, $\angle B = 90^\circ$, $\angle C = 120^\circ$, the point M of intersection of the diagonals satisfies $BM = 1$ and $MD = 2$.

   (a) Prove that the vertices of $ABCD$ lie on a circle and find the radius of that circle.

   (b) Find the area of quadrilateral $ABCD$.

4. Among the n inhabitants of an island, where n is even, every two are either friends or enemies. Some day, the chief of the island orders that each inhabitant (including himself) makes and wears a necklace consisting of marbles, in such a way that two necklaces have a marble of the same type if and only if their owners are friends.

   (a) Show that the chief's order can be achieved by using $n^2/4$ different types of stones.

   (b) Prove that this is not necessarily true with less than $n^2/4$ types.

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