Dutch Mathematical Olympiad 1991

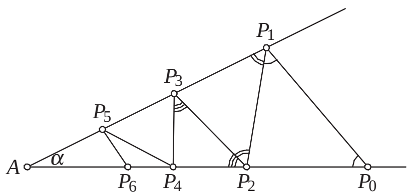
Second Round

September 6

1. Prove that for any threee positive real numbers a, b, c,

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \geq \frac{9}{2} \cdot \frac{1}{a+b+c}.$$

2. An angle with vertex A and measure α and a point P_0 on one of its rays are given so that $AP_0 = 2$. Point P_1 is chosen on the other ray. The sequence of points $P_1, P_2, P_3, \ldots$ is defined so that P_n lies on the segment AP_{n-2} and the triangle $P_nP_{n-1}P_{n-2}$ is isosceles with $P_nP_{n-1} = P_nP_{n-2}$ for all $n \geq 2$.



   (a) Prove that for each value of α there is a unique point P_1 for which the sequence $P_1, P_2, \ldots, P_n, \ldots$ does not terminate.

   (b) Suppose that the sequence $P_1, P_2, \ldots$ does not terminate and that the length of the polygonal line $P_0P_1P_2\ldots P_k$ tends to 5 when $k \to \infty$. Compute the length of P_0P_1.

3. A real function f satisfies $4f(f(x)) - 2f(x) - 3x = 0$ for all real numbers x. Prove that $f(0) = 0$.

4. Three real numbers a, b, c satisfy the equations

$$a+b+c = 3, \quad a^2+b^2+c^2 = 9, \quad a^3+b^3+c^3 = 24.$$

   Find $a^4+b^4+c^4$.

5. Let H be the orthocenter, O the circumcenter, and R the circumradius of an acute-angled triangle ABC. Consider the circles k_a, k_b, k_c, k_h, k, all with radius R, centered at A, B, C, H, M, respectively. Circles k_a and k_b meet at M and F; k_a and k_c meet at M and E; and k_b and k_c meet at M and D.

   (a) Prove that the points D, E, F lie on the circle k_h.

   (b) Prove that the set of points inside k_h that are inside exactly one of the circles k_a, k_b, k_c has the area twice the area of $\triangle ABC$.

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