24-th Nordic Mathematical Contest

April 13, 2010

1. A function $f\colon \mathbb{Z}_+ \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of positive integers, is non-decreasing and satisfies $f(mn) = f(m)f(n)$ for all relatively prime positive integers m and n. Prove that $f(8)f(13) \geq (f(10))^2$.

2. Three circles Γ_A, Γ_B and Γ_C share a common point of intersection O. The other common point of Γ_A and Γ_B is C, that of Γ_A and Γ_C is B, and that of Γ_C and Γ_B is A. The line AO intersects the circle Γ_A in the point $X \neq O$. Similarly, the line BO intersects the circle Γ_B in the point $Y \neq O$, and the line CO intersects the circle Γ_C in the point $Z \neq O$. Show that
$$\frac{|AY|\,|BZ|\,|CX|}{|AZ|\,|BX|\,|CY|} = 1.$$

3. Laura has 2010 lamps connected with 2010 buttons in front of her. For each button, she wants to know the corresponding lamp. In order to do this, she observes which lamps are lit when Richard presses a selection of buttons. (Not pressing anything is also a possible selection.) Richard always presses the buttons simultaneously, so the lamps are lit simultaneously, too.

   a) If Richard chooses the buttons to be pressed, what is the maximum number of different combinations of buttons he can press until Laura can assign the buttons to the lamps correctly?

   b) Supposing that Laura will choose the combinations of buttons to be pressed, what is the minimum number of attempts she has to do until she is able to associate the buttons with the lamps in a correct way?

4. A positive integer is called *simple* if its ordinary decimal representation consists entirely of zeroes and ones. Find the least positive integer k such that each positive integer n can be written as $n = a_1 \pm a_2 \pm a_3 \pm \ldots \pm a_k$ where $a_1, \ldots, a_k$ are simple.

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