23-rd Nordic Mathematical Contest

April 2, 2009

1. A point P is chosen in an arbitrary triangle. Three lines are drawn through P which are parallel to the sides of the triangle. The lines divide the triangle into three smaller triangles and three parallelograms. Let f be the ratio between the total area of the three smaller triangles and the area of the given triangle. Prove that $f \geq \frac{1}{3}$ and determine those points P for which $f = \frac{1}{3}$.

2. On a faded piece of paper it is possible to read the following:

$$(x^2+x+a)(x^{15}-\cdots) = x^{17}+x^{13}+x^5-90x^4+x-90.$$

   Some parts have got lost, partly the constant term of the first factor of the left side, partly the majority of the summands of the second factor. It would be possible to restore the polynomial forming the other factor, but we restrict ourselves to asking the following question: What is the value of the constant term a? We assume that all polynomials in the statement have only integer coefficients.

3. The integers 1, 2, 3, 4, and 5 are written on a blackboard. It is allowed to wipe out two integers a and b and replace them with $a+b$ and ab. Is it possible, by repeating this procedure, to reach a situation where three of the five integers on the blackboard are 2009?

4. 32 competitors participate in a tournament. No two of them are equal and in a one against one mach the better always wins. Show that the gold, silver, and bronze medal winners can be found in 39 matches.