35-th Mongolian Mathematical Olympiad 1999

Final Round – Ulaangom, May 1999

Grade 8

First Day

1. Prove that for any positive integer k there exist infinitely many positive integers m such that $3^k \mid m^3+10$.

2. Let a,b,c be real numbers with $a \geq \frac{8}{5}b > 0$ and $a \geq c > 0$. Prove the inequality
$$\frac{4}{5}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{2}{c} \geq \frac{27}{2}\cdot\frac{1}{a+b+c}.$$

3. Three squares $ABB_1B_2, BCC_1C_2, CAA_1A_2$ are constructed in the exterior of a triangle ABC. In the exterior of these squares, another three squares $A_1B_2B_3B_4$, $B_1C_2C_3C_4$, $C_1A_2A_3A_4$ are constructed. Prove that the area of a triangle with sides C_3A_4, A_3B_4, B_3C_4 is 16 times the area of $\triangle ABC$.

Second Day

4. Is it possible to place a triangle with area 1999 and perimeter 1999^2 in the interior of a triangle with area 2000 and perimeter 2000^2?

5. Real numbers a,b,c satisfy $a^2+b^2+c^2=2$. Prove that
$$|a^3+b^3+c^3-abc| \leq 2\sqrt{2}.$$

6. Two circles in the plane intersect at C and D. A chord AB of the first circle and a chord EF of the second circle pass through a point on the common chord CD. Show that the points A,B,E,F lie on a circle.

Grade 9

First Day

1. The plane is divided into unit cells, and each of the cells is painted in one of two given colors. Find the minimum possible number of cells in a figure consisting of entire cells which contains each of the 16 possible colored 2×2 squares.

2. Find all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying the following conditions

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(i) $f(0) = 1$;

(ii) $f(x + f(y)) = f(x + y) + 1$ for all $x, y \in \mathbb{R}$;

(iii) There exists $x_0 \in \mathbb{Q} \setminus \mathbb{Z}$ such that $f(x_0) \in \mathbb{Z}$.

3. Let M be the centroid of a triangle ABC. Assume that the circumcircle of $\triangle AMC$ is tangent to AB. Prove that

$$\sin\angle CAM + \sin\angle CBM \leq \frac{2}{\sqrt{3}}.$$

Second Day

4. Investigate if there exist infinitely many natural numbers n such that n divides $2^n + 3^n$.

5. Let D be a point in the angle ABC. A circle γ passing through B and D intersects the lines AB and BC at M and N respectively. Find the locus of the midpoint of MN when circle γ varies.

6. Show that there exists a positive integer n such that the decimal representations of 3^n and 7^n both start with the digits 10.

Grade 10

First Day

1. Prove that for any n there exists a positive integer k such that all the numbers $k \cdot 2^s + 1$ ($s = 1, \dots, n$) are composite.

2. The rays $l_1, l_2, \dots, l_{n-1}$ divide a given angle ABC into n equal parts. A line l intersects AB at A_1, BC at A_{n+1}, and l_i at A_{i+1} for $i = 1, \dots, n-1$. Show that the quantity

$$\left(\frac{1}{BA_1} + \frac{1}{BA_{n+1}}\right)\left(\frac{1}{BA_1} + \frac{1}{BA_2} + \cdots + \frac{1}{BA_{n+1}}\right)^{-1}$$

is independent of the line l, and compute its value if $\angle ABC = \varphi$.

3. Does there exist a sequence $(a_n)_{n\in\mathbb{N}}$ of distinct positive integers such that:

(i) $a_n < 1999n$ for all n;

(ii) none of the a_n contains three consecutive decimal digits 1?

Second Day

4. Let p be a prime number and n be a positive integer. Prove that $\varphi(p^n - 1)$ is divisible by n, where φ denotes the Euler function.

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5. Let $A_1,\dots,A_m$ be three-element subsets of an n-element set X such that $|A_i \cap A_j| \le 1$ whenever $i \ne j$. Prove that there exists a subset A of X with $|A| \ge 2\sqrt{n}$ such that it does not contain any of the A_i.

6. A point M lies on the side AC of a triangle ABC. The circle γ with the diameter BM intersects the lines AB and BC at P and Q, respectively. Find the locus of the intersection point of the tangents to γ at P and Q when point M varies.

Teachers – elementary level

First Day

1. In a convex quadrilateral $ABCD$, $\angle ABD = 65^\circ$, $\angle CBD = 35^\circ$, $\angle ADC = 130^\circ$, and $BC = AB$. Find the angles of $ABCD$.

2. Can a square be divided into 10 pairwise non-congruent triangles with the same area?

3. At each vertex of a 4×5 rectangle there is a house. Find the path of the minimum length connecting all these houses.

Second Day

4. A forest grows up p percents during a summer, but gets reduced by x units between two summers. At the beginning of this summer the size of the forest has been a units. How large should x be if we want the forest to increase q times in n years?

5. Find the number of polynomials $P(x)$ of degree 6 whose coefficients are in the set $\{1,2,\dots,1999\}$ and which are divisible by $x^3 + x^2 + x + 1$.

6. Find the minimum possible length of the sum of 1999 unit vectors in the coordinate plane whose both coordinates are nonnegative.

Teachers – secondary level

First Day

1. Suppose that a function $f : \mathbb{R} \to \mathbb{R}$ is such that for any real h there exist at most 19990509 different values of x for which $f(x) \ne f(x+h)$. Prove that there is a set of at most 9995256 real numbers such that f is constant outside of this set.

2. Any two vertices A, B of a regular n-gon are connected by an oriented segment (i.e. either $A \to B$ or $B \to A$). Find the maximum possible number of quadruples (A,B,C,D) of vertices such that $A \to B \to C \to D \to A$.

3. Let $(a_n)_{n=1}^{\infty}$ be a non-decreasing sequence of natural numbers with $a_{20} = 100$. A sequence (b_n) is defined by $b_m = \min\{n \mid a_n \geq m\}$. Find the maximum value of $a_1 + a_2 + \cdots + a_{20} + b_1 + b_2 + \cdots + b_{100}$ over all such sequences (a_n).

Second Day

4. *Problem 4 for Grade 10.*

5. The edge lengths of a tetrahedron are a,b,c,d,e,f, the areas of its faces are S_1,S_2,S_3,S_4, and its volume is V. Prove that
$$2\sqrt{S_1S_2S_3S_4} > 3V\sqrt[6]{abcdef}.$$

6. Let f be a map of the plane into itself with the property that if $d(A,B) = 1$, then $d(f(A), f(B)) = 1$, where $d(X,Y)$ denotes the distance between points X and Y. Prove that for any positive integer n, $d(A,B) = n$ implies $d(f(A), f(B)) = n$.

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