43-th Mongolian Mathematical Olympiad 2007

Final Round

Ulaanbaatar, May 5–11

Grade 11

First Day

1. Let M be the midpoint of the side BC of triangle ABC. The bisector of the exterior angle of point A intersects the side BC in D. Let the circumcircle of triangle ADM intersect the lines AB and AC in E and F respectively. If the midpoint of EF is N, prove that $MN \| AD$.

2. For all $n \geq 2$, let a_n be the product of all coprime natural numbers less than n. Prove that

   (a) $n | a_n + 1 \Leftrightarrow n = 2, 4, p^\alpha, 2p^\alpha$.

   (b) $n | a_n - 1 \Leftrightarrow n \neq 2, 4, p^\alpha, 2p^\alpha$.

   Here p is an odd prime number and $\alpha \in \mathbb{N}$.

3. Let P be a point outside of the triangle ABC in the plane of ABC. Prove that by using reflections S_{AB}, S_{AC}, and S_{BC} across the lines AB, AC, and BC one can shift point P in the triangle ABC.

Second Day

4. If $a, b, c \in \mathbb{R}$ and $a, b, c > 0$ prove that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\sqrt{\frac{a^2 + b^2 + c^2}{ab + bc + ca}}.$$

5. Given a $n \times n$ table with non-negative real entries such that the sums of entries in each column and row are equal, a player plays the following game: The step of the game consists of choosing n cells no two of which share a column or a row, and subtracting the same number from each of the entries of the n cells, provided that the resulting table has all non-negative entries. Prove that the player can change all entries to zeros.

6. Given a quadrilateral $ABCD$ simultaneously inscribed and circumscribed, assume that none of his diagonals or sides is a diameter of the circumscribed circle. Let P be the intersection point of the external bisectors of the angles near A and B. Similarly, let Q be the intersection point of the external bisectors of the angles C and D. If J and O respectively are incenter and circumcenter of $ABCD$ prove that $OJ \perp PQ$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

Teachers – secondary level

First Day

1. Find the number of subsets of the set $\{1,2,3,\dots,5n\}$ such that the sum of the elements in each subset are divisible by 5.
2. Given 101 segments in a line, prove that there exist 11 segments meeting in 1 point or 11 segments such that every two of them are disjoint.
3. Let p be an ood prime number. Let g be primitive root modulo p. Find all the values of p such that the sets $A=\left\{k^2+1 : 1\le k\le \frac{p-1}{2}\right\}$ and $B=\left\{g^m : 1\le m\le \frac{p-1}{2}\right\}$ are equal modulo p.

Second Day

4. If $x,y,z\in\mathbb{N}$ and $xy=z^2+1$ prove that there exist integers a,b,c,d such that $x=a^2+b^2$, $y=c^2+d^2$, $z=ac+bd$.
5. Given a point P in the circumcircle ω of an equilateral triangle ABC, prove that the segments PA, PB, and PC form a triangle T. Let R be the radius of the circumcircle ω and let d be the distance between P and the circumcenter. Find the area of T.
6. Let $n=p_1^{\alpha_1}\cdots p_s^{\alpha_s}\ge 2$. If for any $\alpha\in\mathbb{N}$, $p_i-1\nmid\alpha$, where $i=1,2,\dots,s$, prove that $n\mid\sum_{\alpha\in\mathbb{Z}_n^*}a^{\alpha}$ where $\mathbb{Z}_n^*=\{a\in\mathbb{Z}_n : (a,n)=1\}$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com