38-th Mongolian Mathematical Olympiad 2002

Final Round

Ulaanbaatar, May 10–15

Grade 10

First Day

1. Let n,k be given natural numbers. Find the smallest possible cardinality of a set A with the following property: There exist subsets $A_1, A_2, \dots, A_n$ of A such that the union of any k of them is A, but the union of no $k-1$ of them is A.

2. For a natural number p, one can move between two points with integer coordinates if the distance between them equals p. Find all prime numbers p for which it is possible to reach the point $(2002, 38)$ starting from the origin $(0,0)$.

3. The incircle of a triangle ABC with $AB \neq BC$ touches BC at A_1 and AC at B_1. The segments AA_1 and BB_1 meet the incircle at A_2 and B_2, respectively. Prove that the lines AB, A_1B_1, A_2B_2 are concurrent.

Second Day

4. Let be given 131 distinct natural numbers, each having prime divisors not exceeding 42. Prove that one can choose four of them whose product is a perfect square.

5. Let $a_0, a_1, \dots$ be an infinite sequence of positive numbers. Prove that the inequality $1 + a_n > \sqrt[n]{2} a_{n-1}$ holds for infinitely many positive integers n.

6. Let A_1, B_1, C_1 be the midpoints of the sides BC, CA, AB respectively of a triangle ABC. Points K on segment C_1A_1 and L on segment A_1B_1 are taken such that

$$\frac{C_1K}{KA_1} = \frac{BC + AC}{AC + AB} \quad \text{and} \quad \frac{A_1L}{LB_1} = \frac{AC + AB}{BC + AB}.$$

If BK and CL meet at S, prove that $\angle C_1A_1S = \angle B_1A_1S$.

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Teachers – secondary level

First Day

1. Let D be a point on a semicircle with diameter AB, and C be the midpoint of the arc BD. Let P be an arbitrary point on the ray AD, and Q be a point on AC such that $QP \perp AD$. Prove that the line PQ passes through the intersection of QB and the semicircle. (This is *obviously wrong* - what is the correct formulation?)

2. Prove that for each $n \in \mathbb{N}$ the polynomial $(x^2+x)^{2^n}+1$ is irreducible over the polynomials with integer coefficients.

3. Find all positive integers n for which there exist real numbers $a_1, a_2, \ldots, a_n$ such that
$$\{a_j - a_i \mid 1 \le i < j \le n\} = \left\{1, 2, \ldots, \frac{n(n-1)}{2}\right\}.$$

Second Day

4. Let $p \ge 5$ be a prime number. Prove that there exists $a \in \{1, 2, \ldots, p-2\}$ satisfying $p^2 \nmid a^{p-1} - 1$ and $p^2 \nmid (a+1)^{p-1} - 1$.

5. Let A be the ratio of the product of sides to the product of diagonals in a circumscribed pentagon. Find the maximum possible value of A.

6. Two squares of area 38 are given. Each of the squares is divided into 38 connected pieces of unit area by simple curves. Then the two squares are patched together. Show that one can sting the patched squares by 38 needles so that every piece of each square is stung exactly once.

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