37-th Mongolian Mathematical Olympiad 2001

Final Round

Grade 10

First Day

1. Suppose that a sequence $x_1, x_2, \dots, x_{2001}$ of positive real numbers satisfies
$$3x_{n+1}^2 = 7x_n x_{n+1} - 3x_{n+1} - 2x_n^2 + x_n \quad \text{and} \quad x_{37} = x_{2001}.$$
Find the maximum possible value of x_1.

2. In an acute-angled triangle ABC, a, b, c are sides, m_a, m_b, m_c the corresponding medians, R the circumradius and r the inradius. Prove the inequality
$$\frac{a^2+b^2}{a+b} \cdot \frac{b^2+c^2}{b+c} \cdot \frac{c^2+a^2}{c+a} \geq 16R^2 r \frac{m_a}{a} \cdot \frac{m_b}{b} \cdot \frac{m_c}{c}.$$

3. Let a, b be coprime positive integers with a even and $a > b$. Show that there exist infinitely many pairs (m, n) of coprime positive integers such that $m \mid a^{n-1} - b^{n-1}$ and $n \mid a^{m-1} - b^{m-1}$.

Second Day

4. On a line are given $n > 3$ points. Find the number of colorings of these points in red and blue, such that in any set of consequent points the difference between the numbers of red and blue points does not exceed 2.

5. Let A, B, C, D, E, F be the midpoints of consecutive sides of a hexagon with parallel opposite sides. Prove that the points $AB \cap DE$, $BC \cap EF$, $AC \cap DF$ lie on a line.

6. Some cells of a 10×10 board are marked so that each cell has an even number of neighboring (i.e. sharing a side) marked cells. Find the maximum possible number of marked cells.

Teachers

First Day

1. Prove that for every positive integer n there exists a polynomial $p(x)$ of degree n with real coefficients, having n distinct real roots and satisfying
$$p(x)p(4-x) = p(x(4-x)).$$

2. For positive real numbers $b_1, b_2, \dots, b_n$ define
$$a_1 = \frac{b_1}{b_1+b_2+\cdots+b_n} \quad \text{and} \quad a_k = \frac{b_1+\cdots+b_k}{b_1+\cdots+b_{k-1}} \text{ for } k>1.$$
Prove that $a_1+a_2+\cdots+a_n \le \frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}$.

3. Let $k \ge 0$ be a given integer. Suppose that there exist positive integers n, d and an odd integer $m > 1$ such that $d \mid m^{2^k}-1$ and $m \mid n^d+1$. Find all possible values of $\frac{m^{2^k}-1}{d}$.

Second Day

4. Some cells of a $2n \times 2n$ board are marked so that each cell has an even number of neighboring (i.e. sharing a side) marked cells. Find the number of such markings.

5. Chords AC and BD of a circle w intersect at E. A circle that is internally tangent to w at a point F also touches the segments DE and EC. Prove that the bisector of $\angle AFB$ passes through the incenter of $\triangle AEB$.

6. On a tennis tournament any two of the n participants played a match (the winner of a match gets 1 point, the loser gets 0). The scores after the tournament were $r_1 \le r_2 \le \cdots \le r_n$. A match between two players is called *wrong* if after it the winner has a score less than or equal to that of the loser. Consider the set $I = \{i \mid r_i \ge i\}$. Prove that the number of wrong matches is not less than $\sum_{i\in I}(r_i - i + 1)$, and show that this value is realizable.

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