14-th Mexican Mathematical Olympiad 2000.

Morelia, Michoacán

First Day

1. Circles A,B,C,D are given on the plane such that circles A and B are externally tangent at P, B and C at Q, C and D at R, and D and A at S. Circles A and C do not meet, and so do not B and D.

   (a) Prove that the points P,Q,R,S lie on a circle.

   (b) Suppose that A and C have radius 2, B and D have radius 3, and the distance between the centers of A and C is 6. Compute the area of the quadrilateral $PQRS$.

2. A triangle of numbers is constructed as follows. The first row consists of the numbers from 1 to 2000 in increasing order, and under any two consecutive numbers their sum is written. (See the example corresponding to 5 instead of 2000 below.) What is the number in the lowermost row?

$$\begin{array}{ccccccccc}
1 & & 2 & & 3 & & 4 & & 5 \\
 & 3 & & 5 & & 7 & & 9 & \\
 & & 8 & & 12 & & 16 & & \\
 & & & 20 & & 28 & & & \\
 & & & & 48 & & & &
\end{array}$$

3. Given a set A of positive integers, the set A' is composed from the elements of A and all positive integers that can be obtained in the following way: Write down some elements of A one after another without repeating, write a sign $+$ or $-$ before each of them, and evaluate the obtained expression; The result is included in A'. For example, if $A = \{2,8,13,20\}$, numbers 8 and $14 = 20-2+8$ are elements of A'. Set A'' is constructed from A' in the same manner. Find the smallest possible number of elements of A, if A'' contains all the integers from 1 to 40.

Second Day

4. Let a and b be positive integers not divisible by 5. A sequence of integers is constructed as follows: the first term is 5, and every consequent term is obtained by multiplying its precedent by a and adding b. (For example, if $a = 2$ and $b = 4$, the first three terms are $5,14,32$.) What is the maximum possible number of primes that can occur before encoutering the first composite term?

5. A board $n \times n$ is colored black and white like a chessboard. The following steps are permitted: Choose a rectangle inside the board (consisting of entire cells) whose side lengths are both odd or both even, but not both equal to 1, and invert the colors of all cells inside the rectangle. Determine the values of n for which

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it is possible to make all the cells have the same color in a finite number of such steps.

6. Let ABC be a triangle with $\angle B > 90^\circ$ such that there is a point H on side AC with $AH = BH$ and BH perpendicular to BC. Let D and E be the midpoints of AB and BC respectively. A line through H parallel to AB cuts DE at F. Prove that $\angle BCF = \angle ACD$.

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