7-th Mediterranean Mathematical Competition 2004

1. Find all natural numbers m such that
$$1! \cdot 3! \cdot 5! \cdots (2m-1)! = \left(\frac{m(m+1)}{2}\right)!.$$

2. In a triangle ABC, the altitude from A meets the circumcircle again at T. Let O be the circumcenter. The lines OA and OT intersect the side BC at Q and M, respectively. Prove that
$$\frac{S_{AQC}}{S_{CMT}} = \left(\frac{\sin B}{\cos C}\right)^2.$$

3. Prove that if a,b,c are positive numbers satisfying $1 = ab+bc+ca+2abc$, then
$$2(a+b+c)+1 \geq 32abc.$$

4. Let z_1,z_2,z_3 be pairwise distinct complex numbers satisfying $|z_1| = |z_2| = |z_3| = 1$ and
$$\frac{1}{2+|z_1+z_2|} + \frac{1}{2+|z_2+z_3|} + \frac{1}{2+|z_3+z_1|} = 1.$$
If the points $A(z_1), B(z_2), C(z_3)$ are vertices of an acute-angled triangle, prove that this triangle is equilateral.

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