4-th Mediterranean Mathematical Competition 2001

1. Let \( P \) and \( Q \) be points on a circle \( k \). A chord \( AC \) of \( k \) passes through the midpoint \( M \) of \( PQ \). Consider a trapezoid \( ABCD \) inscribed in \( k \) with \( AB \parallel CD \). Prove that the intersection point \( X \) of \( AD \) and \( BC \) depends only on \( k \) and \( P,Q \).

2. Find all integers \( n \) for which the polynomial \( p(x) = x^5 - nx - n - 2 \) can be represented as a product of two non-constant polynomials with integer coefficients.

3. Show that there exists a positive integer \( N \) such that the decimal representation of \( 2000^N \) starts with the digits 200120012001.

4. Let \( \mathcal{S} \) be the set of points inside a given equilateral triangle \( ABC \) with side 1 or on its boundary. For any \( M \in \mathcal{S} \), \( a_M, b_M, c_M \) denote the distances from \( M \) to \( BC, CA, AB \), respectively. Define
   \[
   f(M) = a_M^3(b_M - c_M) + b_M^3(c_M - a_M) + c_M^3(a_M - b_M).
   \]
   (a) Describe the set \( \{ M \in \mathcal{S} \mid f(M) \geq 0 \} \) geometrically.
   (b) Find the minimum and maximum values of \( f(M) \) as well as the points in which these are attained.