Moldovan Team Selection Tests 2002

First Test

March 13

1. Consider the triangular numbers $T_n = \frac{n(n+1)}{2}$, $n \in \mathbb{N}$.

   (a) If a_n is the last digit of T_n, show that the sequence (a_n) is periodic and find its basic period.

   (b) If s_n is the sum of the first n terms of the sequence (T_n), prove that for every $n \geq 3$ there is at least one perfect square between s_{n-1} and s_n.

2. Prove that there exists a partition of the set $A = \{1^3, 2^3, \dots, 2000^3\}$ into 19 nonempty subsets such that the sum of elements of each subset is divisible by 2001^2.

3. The circles $\Gamma_1(O_1)$, $\Gamma_2(O_2)$, $\Gamma_3(O_3)$ are such that Γ_1 and Γ_2 are externally tangent at A, Γ_2, Γ_3 are so at B, and Γ_3, Γ_1 are so at C. Let A_1 and B_1 be points on Γ_1 diametrically opposite to A and B respectively, and let AB_1 meet Γ_2 again at M, BA_1 meet Γ_3 again at N, and AA_1 and BB_1 meet at P. Prove that points M, N, P are collinear.

4. The sequence $P_n(x)$, $n \in \mathbb{N}$ of polynomials is defined as follows:

$$P_0(x) = x, \quad P_1(x) = 4x^3 + 3x,$$
$$P_{n+1}(x) = (4x^2 + 2)P_n(x) - P_{n-1}(x) \quad \text{for } n \geq 1.$$

   For every positive integer m, we consider the set $A(m) = \{P_n(m) \mid n \in \mathbb{N}\}$. Show that the sets $A(m)$ and $A(m+4)$ have no common elements.

Second Test

April 6

1. Positive numbers $\alpha, \beta, x_1, x_2, \dots, x_n$ $(n \geq 1)$ satisfy $x_1 + x_2 + \cdots + x_n = 1$. Prove that

$$\frac{x_1^3}{\alpha x_1 + \beta x_2} + \frac{x_2^3}{\alpha x_2 + \beta x_3} + \cdots + \frac{x_n^3}{\alpha x_n + \beta x_1} \geq \frac{1}{n(\alpha + \beta)}.$$

2. Let A be a set containing $4k$ consecutive positive integers, where $k \geq 1$ is an integer. Find the smallest k for which the set A can be partitioned into two subsets having the same number of elements, the same sum of elements, the same sum of the squares of elements, and the same sum of the cubes of elements.

3. A triangle ABC is inscribed in a circle Γ. Points M and N are the midpoints of the arcs BC and AC respectively, and D is an arbitrary point on the arc AB (not containing C). Points I_1 and I_2 are the incenters of the triangles ADC and BDC, respectively. If the circumcircle of triangle DI_1I_2 meets Γ again at P, prove that triangles PNI_1 and PMI_2 are similar.

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4. Let C be the circle with center $O(0,0)$ and radius 1, and $A(1,0)$, $B(0,1)$ be points on the circle. Distinct points $A_1, A_2, \dots, A_{n-1}$ on C divide the smaller arc AB into n equal parts ($n \geq 2$). If P_i is the orthogonal projection of A_i on OA ($i = 1, \dots, n-1$), find all values of n such that $P_1A_1^{2p} + P_2A_2^{2p} + \cdots + P_{n-1}A_{n-1}^{2p}$ is an integer for every positive integer p.

Third Test

April 7

1. Prove that for every integer $n \geq 1$ there exists a polynomial $P(x)$ with integer coefficients such that $P(1), P(2), \dots, P(n)$ are distinct powers of two.

2. Let $A = \{a_1, a_2, \dots, a_n\}$ be a set of $n \geq 1$ positive real numbers. For each nonempty subset of A the sum of its elements is written down. Show that all written numbers can be divided into n classes such that in each class the ratio of the greatest number to the smallest number is not greater than 2.

3. A triangle ABC is inscribed in a circle Γ. For any point M inside the triangle, A_1 denotes the intersection of the ray AM with Γ. Find the locus of point M for which $\dfrac{BM \cdot CM}{MA_1}$ is minimal, and find this minimum value.

4. Let $P(x)$ be a polynomial with integer coefficients for which there exists a positive integer n such that the real parts of all roots of $P(x)$ are less than $n - \frac{1}{2}$, polynomial $x - n + 1$ does not divide $P(x)$, and $P(n)$ is a prime number. Prove that the polynomial $P(x)$ is irreducible (over $\mathbb{Z}[x]$).