42-nd Moldova Mathematical Olympiad 1998

Final Round – Chişinău, April 3–4

Grade 7

First Day

1. Initially there are 10 empty sacks. In each step we put 9 new empty sacks in one of the empty sacks. After several steps we obtain 221 nonempty sacks. How many sacks are there in total?

2. Find all four-digit natural numbers n with the sum of digits equal to $2027-n$.

3. Find all real solutions $x_1, x_2, \dots, x_{1999}$ of the system
$$1+x_1^2 = 2x_2, \quad 1+x_2^2 = 2x_3, \quad \dots, \quad 1+x_{1999}^2 = 2x_1.$$

4. Point N is taken on the median BM of a triangle ABC such that $AN = BC$. Line AN meets BC at K. Prove that $BK = KN$.

Second Day

5. Păcală produces wine of concentration $P\%$ and fills up a cask in P hours. Tândală produces wine of concentration $T\%$ and fills up the same cask in T hours. Working together, they fill up the cask in 24 hours. What is the concentration of wine in the cask prepared in this manner?

6. The numbers $1,2,\dots,1999$ are written in the squares of a 1999×1999 table such that every two numbers in the same row or column are distinct. In every two squares that are symmetric with respect to the main diagonal, the written numbers are equal. Prove that all numbers on the main diagonal are distinct.

7. If a,b,c are positive numbers, prove the inequality
$$\frac{a^2}{a^2+2bc}+\frac{b^2}{b^2+2ca}+\frac{c^2}{c^2+2ab} \geq 1.$$

8. Let P be an arbitrary point on the base BC of an isosceles triangle ABC. Let D and K be the projections of P onto AB and AC respectively, and let M and N be the points on the segments AD and AK respectively such that $2MD = MB$ and $2NK = NC$. Prove that $MC = NB$.

Grades 8 and 9

First Day

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1. One hundred arbitrary natural numbers are arranged in a sequence. Show that it is always possible to choose several consecutive terms of the sequence such that their sum is divisible by 100.

2. Prove that if real numbers a,b,c,d verify the inequality

$$(a+b+c+d)^2 \geq 2(a^2+b^2+c^2+d^2),$$

then the equation $(a-x)(b-x)+(c-x)(d-x)=0$ has real roots.

3. Every inhabitant of the island Tau paints himself in one of three colors: in red, yellow, or blue if he has one, two, three friends, respectively. No inhabitant painted in blue has a friend painted in yellow. Some day, 400 inhabitants painted in blue and 33 painted in yellow repaint themselves into red color, while 225 inhabitants painted in red change the color to blue. As a result, every two friends are of the same color. How many inhabitants does the island Tau have?

4. The diagonals of a convex quadrilateral intersect at point O. Points K and E are taken on side BC such that $BK=KE=EC$. The rays KO and CD meet at N and the rays EO and BA meet at N, where $CN=2DN$ and $BM=2AB$. Prove that $BC=MN$.

Second Day

5. Prove that the difference of the product of the positive even numbers not exceeding 2000 and the product of the positive odd numbers not exceeding 1999 is divisible by 2001.

6. Some squares of a 1999×1999 table are colored in such a way that every row or column contains exactly one colored square. Show that every square 1000×1000 with the sides going along the grid lines contains at least one colored square.

7. If a,b,c are positive numbers, prove the inequality

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)} \geq \frac{a}{c+a}+\frac{b}{a+b}+\frac{c}{b+c}\,.$$

8. In a right triangle ABC with the right angle at A, a circle is tangent to the median AD at A. The circle intersects the line BC at M and N. Prove that one leg of the triangle bisects the angle MAN.

Grade 10

First Day

1. Consider the function $f(x)=x^2-2ax-a^2-\frac{3}{4}$. Find the values of a such that $f(x)\leq 1$ holds for all $x\in[0,1]$.

2. A positive integer n is such that the numbers $2n^2$ and $3n^2$ have 28 and 30 distinct divisors, respectively. How many divisors does the number $6n^2$ have?

3. The natural numbers from 1 to 100 are arranged around a circle. Prove that among the sums of two adjacent numbers there exist two which differ by more than 2.

4. Let CD be the altitude of a triangle ABC with $AB = 1999$, $BC = 1998$ and $AC = 2000$. The incircles of triangles ACD and BCD are tangent to CD at M and N, respectively. Find the length of MN.

Second Day

5. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) = [x]f(\{x\}) + \{x\}f([x]) \quad \text{for all } x,$$

where $[x]$ and $\{x\}$ denote the integral and fractional part of x.

6. Find a cubic polynomial with real coefficients whose roots are the squares of the roots of the polynomial $P(x) = x^3 + 9x^2 + 9x + 9$.

7. Prove that if a, b, c are positive numbers and $x = \sqrt[3]{abc}$, then

$$\frac{1}{a+b+x} + \frac{1}{b+c+x} + \frac{1}{c+a+x} \leq \frac{1}{x}.$$

8. Points D and E are taken on the sides BC and AB respectively of an equilateral triangle ABC such that $CD/DB = BE/EA = \frac{\sqrt{5}+1}{2}$. The lines AD and CE intersect at O. Let M and N be the points on the segments OD and OC, respectively, such that $MN \parallel BC$ and $AN = 2OM$. The line through O parallel to AC meets the segment MC at P. Prove that the ray AP bisects the angle MAN.

Grades 11 and 12

First Day

1. Grandfather arranges n grandsons around a circle and gives them n sweets as follows. He chooses one of his grandchildren arbitrarily and gives him a sweet. Thereafter, going in a fixed direction around the circle, he gives a sweet to the first grandson, then he skips one and gives a sweet to the next grandson, then he skips two and gives a sweet to the next grandson, and so on. For which values of n does each grandchild get a sweet?

2. Given a positive integer n, let M be the set of all numbers x of the form

$$x = \frac{a_1}{2} + \frac{a_2}{2^2} + \cdots + \frac{a_n}{2^n}, \quad a_i \in \{0, 1\}.$$

(a) Determine the set M and show that the sequence (a_i) is unique for every $x \in M$.

(b) Find the function $f : M \to \mathbb{R}$ with the property that

$$f(x) = \frac{2000^{a_1}}{2} + \frac{2000^{a_2}}{2^2} + \cdots + \frac{2000^{a_n}}{2^n} \quad \text{for all } x \in M,$$

where the sequence (a_i) is defined as above.

3. Find the smallest value of $E(x,y) = \sqrt{3+x} + \sqrt{3-y}$, where x and y are real numbers satisfying $x^2 + y^2 = 9$.

4. Prove that if all faces of a tetrahedron are congruent triangles, then these triangles are acute-angled.

Second Day

5. Find all integer values of the parameter m for which the equation

$$\left[\frac{m^2 x - 13}{1999}\right] = \frac{x - 12}{2000}$$

has 1999 distinct real solutions.

6. Initially the number $99\ldots9$ ($n \geq 2$ nine's) is written on the board. Each minute, one selects a number written on the board, decomposes it into a product of two positive integer factors, independently increases or decreases each factor by 2 (so that no negative numbers are obtained), and replaces the selected number by the two obtained numbers. For which values of n is it possible to achieve in finite time that all numbers on the board are equal to 9?

7. Prove that the inequality $a^m + a^n \geq m^m + n^n$ holds for all positive integers m and n, where

$$a = \frac{m^{m+1} + n^{n+1}}{m^m + n^n}.$$

8. Let M, N, and P be the points on the sides Bc, CA, AB respectively of an equilateral triangle ABC such that $BM/MC = CN/NA = AP/PB = \lambda$. Find all values of λ for which the (possibly degenerate) triangle bounded by the lines AM, BN, and CP lies entirely within the circle with diameter AC.