41-st Moldova Mathematical Olympiad 1997

Final Round – Chişinău, March 20–21

Grade 7

First Day

1. Find the greatest natural number d that divides every number of the form $n(n+1)(2n+1)$, where n is a positive integer.
2. The side lengths of a triangle are consecutive integers. Find the side lengths if it is known that a median of this triangle is perpendicular to an angle bisector.
3. A square can be divided into 25 smaller squares, all but one having a side of length 1. Find the area of this square.
4. A subset M of the set $A = \{1, 2, \dots, 50\}$ has the property that the sum of any two distinct elements of M is not divisible by 7. Find the maximum possible number s of elements of M.

Second Day

5. Consider 1997 consecutive even natural numbers. Prove that among any 41 of them there exist two whose absolute difference is less than 100.
6. In a parallelogram $ABCD$, E is the midpoint of AD and F is the foot of the perpendicular from B to CE. Prove that the triangle ABF is isosceles.
7. Let a and b be positive integers. Show that $a^2 + b^2$ is divisible by ab if and only if $a = b$.
8. We are given 25 pieces of cheese. Is it always possible to cut one of the pieces into two parts and to pack up all the cheese in two packets, so that the cut parts are in different packets and both packets have the same number of pieces and the same weight?

Grade 8

First Day

1. *Problem 3 for Grade 7.*
2. Find the greatest natural number d that divides every number of the form $n(n+1)(2n+1996)$, where n is a positive integer.

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3. In a quadrilateral $ABCD$ the angles at A and C are equal. The bisector of the angle at B meets the circumcircle of the triangle BDC at point $C_1 \neq D$, and the bisector of the angle at D meets the circumcircle of the triangle BDA at point $A_1 \neq B$. Prove that the quadrilateral A_1BC_1D is a parallelogram.

4. A subset M of the set $A = \{1,2,\dots,547\}$ has the property that the sum of any two distinct elements of M is not divisible by 42. Find the maximum possible number s of elements of M.

Second Day

5. Let $(a_1,a_2,\dots,a_n)$ and $(b_1,b_2,\dots,b_n)$ be two permutations of the numbers $1,2,\dots,n$. Prove that if n is even, then among the numbers a_i+b_i, $i = 1,2,\dots,n$, there always exist two leaving the same remainder when divided by n.

6. Find all prime numbers of the form $\overline{10101\dots01}$ in the decimal system.

7. An acute angle with the vertex A and a point P inside it are given. Show how to construct a line through P intersecting the rays of the angle at B and C such that $\frac{1}{BP}+\frac{1}{CP}$ is minimum.

8. We are given 1997 pieces of cheese. Is it always possible to cut one of the pieces into two parts and to pack up all the cheese in two packets, so that the cut parts are in different packets and both packets have the same number of pieces and the same weight?

Grade 9

First Day

1. Find all positive integers n for which $n+200$ and $n-269$ are both cubes of positive integers.

2. Let $a > 1$ be an integer and let M be the set of remainders obtained when a is divided by all positive integers less than a. It is known that the sum of the (distinct) elements of M equals a. Determine a.

3. Prove that if $a_1,a_2,\dots,a_n$ are positive numbers with $a_1a_2\cdots a_n = 1$, then

$$\sqrt{a_1}+\sqrt{a_2}+\cdots+\sqrt{a_n} \leq a_1+a_2+\cdots+a_n.$$

4. Two circles of radii R and r are tangent to a line l at points A and B and intersect each other at C and D. Prove that the circumradii of the triangles ABC and ABD are equal and compute them.

Second Day

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5. Let S be the area of a convex quadrilateral $ABCD$. Points A_1, B_1, C_1, and D_1 are taken on the rays AB, BC, CD, DA such that $AA_1 = 2AB$, $BB_1 = 2BC$, $CC_1 = 2CD$, $DD_1 = 2DA$. Determine the area of the quadrilaterals $A_1B_1C_1D_1$.

6. Consider the fraction $0.a_1a_2a_3\dots$, where the (decimal) digits a_1 and a_2 are arbitrary and every following digit equals the remainder of the sum of the previous two upon division by 10. Prove that the fraction is periodic.

7. Suppose that among any four participants of a competition there is at least one who knows the other three. Prove that there is a participant who knows all the others.

8. In a triangle ABC with $AB = BC$ and $\angle B = 80^\circ$, P is the interior point such that $\angle PAC = 40^\circ$ and $\angle ACP = 30^\circ$. Determine $\angle BPC$.

Grade 10

First Day

1. Find the values of a parameter p for which the equation

$$x^4 - (3p+2)x^2 + p^2 = 0$$

has four real solutions forming an arithmetic progression.

2. Find the sum of digits of the sum of digits of the sum of digits of 1997^{1997}.

3. Let $\mathscr{C}$ be the circumcircle of an acute triangle XYZ. For a point P inside $\mathscr{C}$ denote by $PX \cap \mathscr{C} = \{L\}$, $PY \cap \mathscr{C} = \{M\}$, and $PZ \cap \mathscr{C} = \{N\}$. Find the locus of P for which triangle LMN is equilateral.

4. The sequences (a_n) and (b_n) are defined by $a_1 = 9$, $b_1 = 3$ and

$$a_{k+1} = 9^{a_k}, \quad b_{n+1} = 3^{b_n}, \quad \text{for all } k \in \mathbb{N}.$$

Find the least n for which $b_n > a_{1997}$.

Second Day

5. Two real numbers a and b satisfy the relations

$$a^3 - 3ab^2 = 29 \quad \text{and} \quad b^3 - 3a^2b = 34.$$

Compute $a^2 + b^2$.

6. The teacher has a number of candies and wants to distribute them among 13 boys and 10 girls so that all boys receive the same number of candies (at least one) and all girls receive the same number of candies (at least one). If he can do this in a unique way, find the greatest possible number of candies he can have.

7. Let $ABCD$ be a unit square and let X and Y be arbitrary points on the sides AB and CD, respectively. The lines XD and YA meet at M and the lines XC and YB meet at N. For which X and Y is the area of the quadrilateral $XNYM$ largest possible? Find this area.

8. Define $a_n = \left[\frac{n}{1}\right] + \left[\frac{n}{2}\right] + \cdots + \left[\frac{n}{n}\right]$, where n is a natural number. Prove that $a_n = 2 + a_{n-1}$ if and only if n is a prime number.

Grades 11 and 12

First Day

1. A cube can be divided into 99 smaller cubes, all but one having a side of length 1. Find the volume of this cube.

2. Find the first 1997 decimal digits of the number $\left(7+\sqrt{50}\right)^{1997}$.

3. All faces of a tetrahedron have equal area and all its edges are tangent to a single sphere. Prove that the tetrahedron is regular and compute the ratio of the volumes of the sphere and the tetrahedron.

4. Consider the set $A = \{1, 2, \dots, n\}$, where $n \geq 2$ is a given integer. Denote by $S(n)$ the greatest possible cardinality of a subset M of A such that the sum of no two elements of M is divisible by 42. Given that $S(n) = 1997$, find n.

Second Day

5. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy

   (a) $f(n+m) = f(n)f(m)$ for all $n, m \in \mathbb{N}$, and

   (b) the equation $f(f(x)) = f(x)^2$ has at least one solution in $\mathbb{N}$.

6. Two right cones with the common vertex S are inscribed in a sphere. An arbitrary plane α containing point S intersect the bases of the two cones in the segments AB and CD. Prove that the product of a diagonal and a lateral side in the trapezoid with the bases AB and CD is independent of the choice of α.

7. A polynomial $P(x)$ of degree $n \geq 5$ with integer coefficients has n distinct integer roots $\alpha_1 = 0, \alpha_2, \dots, \alpha_n$. Find all integer roots of the polynomial $P(P(x))$.

8. Let n coins be put into several (maybe one) piles. What is the greatest possible value of the product of the numbers of coins in the piles?

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