40-th Moldova Mathematical Olympiad 1996

Final Round

Grade 7

First Day

1. The three-digit number $\overline{abc}$ is divisible by 37. Show that so is the number $\overline{bca}+\overline{cab}$.

2. Find two distinct rational numbers with the denominators 8 and 13 such that the difference between them is as small as possible.

3. In an isosceles triangle ABC, AD is the altitude to the base BC, and M a point on side AC. Prove that $DB-DM < AB-AM$.

4. Two friends got employed and started with the job at January 1, 1996. The first one has one free day after three consecutive working days, while the second one has three free days after seven working days. Determine when the two friends will have the first free day in common in year 2000.

Second Day

5. Numbers $y_1,y_2,\dots,y_{25}$ are a permutation of integers $x_1,x_2,\dots,x_{25}$. Show that the product $(x_1-y_1)(x_2-y_2)\cdots(x_n-y_n)$ is even.

6. Find the largest possible value of the ratio of a three-digit number to its sum of digits.

7. A line b through the vertex A of a triangle ABC is perpendicular to the bisector AD of the angle A. Find the point M on line b that minimizes the perimeter of the triangle MBC.

8. Fourty children attend a primary school in a village. Every two children have a common grandfather. Show that at least one grandfather has at least 21 grandchildren in the school.

Grade 8

First Day

1. How many nine-digit numbers are there which end with 1996 and are divisible by 1996?

2. A polynomial $f(x)=x^n+a_1x^{n-1}+\cdots+a_n$ with rational coefficients has a rational root r. Show that r is an integer and that for every integer m, $f(m)$ is a multiple of $r-m$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

3. Two circles intersect at M and N. On the first circle is chosen a point A, different from M and N. Lines AM and AN meet the second circle again at B and C, respectively. Prove that the tangent to the first circle at A is parallel to BC.

4. There are $2n+1$ given objects. There are M ways of selecting a set containing an odd number of objects, and N ways of selecting a set containing an even number of objects. Show that $M = N$.

Second Day

5. Let x and y be positive integers with $xy = 1995^{1996}$. Prove that $x+y$ is not a multiple of 1996.

6. Find all real solutions of the equation $x - [x] = [\frac{1}{2}x - 2]$.

7. In the plane are given $2n$ points. Show that there is a non-intersecting polygonal line with vertices at these points.

8. A chess player won 40 points in 100 matches. What is the difference between the numbers of victories and defeats of that player?

Grade 9

First Day

1. By deleting the last (decimal) digit, a natural number is decreased by 1996. Find all such numbers.

2. Prove that $(1+a+a^2+a^3)^2 \leq 4(1+a^2+a^4+a^6)$ for all real a.

3. Let B and B_1 be points on side AK of a triangle ADK, with B_1 between A and B. The lines through B and B_1 parallel to AD meet DK at C and C_1, respectively. The median KN of triangle ADK intersects BC at M. Show that the lines AC, NC_1, and B_1M are either concurrent or parallel.

4. Each of the three referees gives each of the ten skaters a mark from 1 to 10 points. It is known that only one skater got the smallest sum of points. Which is the largest possible value of this smallest sum, assuming that each referee gave each of the marks 1 to 10?

Second Day

5. Let n be a natural number. Prove that $2^n + n^2$ is a multiple of 5 if and only if so is $n^2 \cdot 2^n + 1$.

6. Find all real solutions of the equation

$$x^2 - \frac{1}{2}[x^2 - 2] = 2 + 3\left[\frac{1}{2}x^2 - 3\right].$$

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com

7. Three circles k_1, k_2, k_3 are tangent to each other at three distinct points. We say that points A and B on two different circles (distinct from their tangency point) are *correspondent* if the line joining them contains the tangency point. Let $A \in k_1$, $B \in k_2$, $C \in k_3$ and $D \in k_1$ be such that A and B, B and C, C and D are correspondent. Show that AD is a diameter of k_1.

8. In a factory, every regular tetrahedron is colored with the four given colors, each face with a different color. How many different colorings can we have?

Grade 10

First Day

1. Let $n = 2^{13} \cdot 3^{11} \cdot 5^7$. Find the number of divisors of n^2 which are less than n and do not divide n.

2. Different quadratic trinomials $f(x)$ and $g(x)$ are monic and satisfy

$$f(-12) + f(2000) + f(4000) = g(-12) + g(2000) + g(4000).$$

Find all real numbers x satisfying the equation $f(x) = g(x)$.

3. Through the vertices of a given triangle tangents to its circumcircle have been drawn. The distances from an arbitrary point to the vertices of the triangle are a, b, c, and the distances to the tangents x, y, z. Prove that $a^2 + b^2 + c^2 = xy + yz + zx$.

4. Two brothers have sold n lambs at the price of n dollars each. The brothers divided the income between themselves in the following way: the elder brother took 10 dolars, then the younger took 10 dollars, then the elder took another 10 dollars, and so on. At the end, there were less than 10 dollars for the younger brother, so he took a pocket-knife from the elder brother, thus making the division fair. What was the value of the pocket knife?

Second Day

5. Prove that for any integers $m, n \geq 2$, at least one of the numbers $\sqrt[m]{n}$ and $\sqrt[n]{m}$ is not greater than $\sqrt[3]{3}$.

6. If $a_1, a_2, \dots, a_{1996}$ are nonpositive real numbers, prove the inequality

$$2^{a_1} + 2^{a_2} + \cdots + 2^{a_{1996}} \leq 1995 + 2^{a_1 + a_2 + \cdots + a_{1996}}.$$

7. In a triangle ABC, the perpendicular bisectors of sides BC and AC intersect lines AC and BC respectively at M and N. Let O be the circumcenter of triangle ABC. Prove that:

   (a) Points A, B, M, N, O lie on a single circle k;

(b) The radius of circle k is equal to the circumradius of triangle MNC.

8. In a pile of 1996 apparently equal coins two are fakes, one being heavier and one being lighter then a regular coin. Using a balance without scales, determine in at most four measurings whether the mass of the two fakes is less than, equal to, or more than the mass of two regular coins.

Grades 11 and 12

First Day

1. Prove the equality
$$\frac{1}{666}+\frac{1}{667}+\cdots+\frac{1}{1996}=1+\frac{2}{2\cdot3\cdot4}+\frac{2}{5\cdot6\cdot7}+\cdots+\frac{2}{1994\cdot1995\cdot1996}.$$

2. Prove that the product of the solutions of the equation $\sqrt{1996}x^{\log_{1996}x}=x^6$ is an integer and find the last four digits of this integer.

3. Two disjoint circles k_1,k_2 with centers O_1 and O_2, respectively, are given in the plane. A common external tangent touches k_1 at A and k_2 at B. Line O_1O_2 meets k_1 and k_2 at C and D, respectively. Prove that

   (a) points A,B,C,D are concyclic;

   (b) lines AC and BD are perpendicular.

4. Among n apparently equal coins, less than a half are fakes differing from the regular coins by weight. Prove that it is possible to find at least one regular coin in at most $n-1$ measurings with a balance without scales.

Second Day

5. Let p be the number of functions from the set $\{1,2,\dots,m\}$ to $\{1,2,\dots,36\}$ (where $m\in\mathbb{N}$), and let q be the number of functions from the set $\{1,2,\dots,n\}$ to $\{1,2,3,4,5\}$ (where $n\in\mathbb{N}$). Find the smallest possible value of $|p-q|$.

6. Solve the equation $2x^2-3x=2x\sqrt{x^2-3x}+1$ in the set of real numbers.

7. Different points A,B,C,D,E on a sphere are such that the segments AB and CD intersect at F and points A,C,F are equidistant from point E. Prove that the lines BD and EF are perpendicular.

8. Twenty children attend a primary school in a village. Every two children have a common grandfather. Show that at least one grandfather has at least 14 grandchildren in the school.