46-th Moldova Mathematical Olympiad 2002

Final Round – Chişinău, March 11–12

Grade 7

First Day

1. Seven pupils on a vacation decided to send postcards to each other, each sending a postcard to exactly three other pupils. Is it possible that each pupil receives postcards exactly from those who receive his postcards?

2. Real numbers a,b,c satisfy $a+b+c=1$. Prove the inequality
$$a^2+b^2+c^2 \geq 4(ab+ac+bc)-1.$$

3. An acute angle DEF and points B and C on the ray EF are given. Find the position of point A on the ray ED for which the sum $AB+AC$ is minimal.

4. Twelve teams participated on a soccer tournament. According to the rules, a team gets 2 points for a victory, 1 for a draw, and 0 for a defeat. When the tournament was over, all teams had distinct numbers of points, and the team that ranked second had as many points as the teams on the last five places in total. Who won the match between the fourth and eighth place teams?

Second Day

5. Volume A equals one fourth of the sum of volumes B and C, while volume B equals one sixth of the sum of volumes A and C. Find the ratio of volume C to the sum of volumes A and B.

6. Five parcels of land are given. In each step, we divide one parcel into three or four smaller ones. Assume that, after several steps, the number of obtained parcels equals four times the number of steps. How many steps were performed?

7. In a triangle ABC, the angle bisector at B intersects AC at D and the circumcircle again at E. The circumcircle of the triangle DEA meets the segment AB again at F. Show that the triangles DBC and DBF are congruent.

8. Find the number of triples (a,b,c) of integers satisfying the equalities
$$\frac{2a-b+6}{4a+c+2}=\frac{b-2c}{a-c}=\frac{2a+b+2c-2}{6a+2c-2}.$$

Grade 8

First Day

The IMO Compendium Group,
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1. Find all real solutions of the equation $[x] + \left[x + \frac{1}{2}\right] + \left[x + \frac{2}{3}\right] = 2002$.

2. Given a positive integer k, find all positive integers n with the property that one can obtain the sum of the first n positive integers by writing some k digits to the right of n (in the decimal system).

3. Let P be a given point inside the circle $C(O,R)$. For every chord AB of the circle passing through P, let t_1 and t_2 be the tangents to the circle at A and B respectively, and M and N be the feet of the perpendiculars from P to t_1 and t_2. Prove that the quantity $\frac{1}{PM} + \frac{1}{PN}$ is independent of the choice of chord AB.

4. All internal phone numbers in a certain company have four digits. The director wants the phone numbers of the administration offices to consist of digits 1,2,3 only, and that any two of these phone numbers coincide in at most one position. How many distinct phone numbers can these offices have?

Second Day

5. Several pupils wrote a solution of a math problem on the blackboard on the break. When the teacher came in, a pupil was just clearing the blackboard, so the teacher could only observe that there was a rectangle with the sides of integer lengths and a diagonal of length 2002. Then the teacher pointed out there was a computation error in the pupils' solution. How did he conclude that?

6. From a set of consecutive natural numbers one number is excluded so that the arithmetic mean of the remaining numbers is 50.55. Find the initial set of numbers and the excluded number.

7. In a triangle ABC, the bisectors of the angles at B and C meet the opposite sides at B_1 and C_1, respectively. Let T be the midpoint of AB_1. Lines BT and B_1C_1 meet at E and lines AB and CE meet at L. Prove that the lines TL and B_1C_1 have a point in common.

8. Find the maximum and minimum values of the expression

$$E = \frac{(1+x)^8 + 16x^4}{(1+x^2)^4}, \quad x \in \mathbb{R}.$$

Grade 9

First Day

1. Real numbers $a \neq 0$, b, c are such that the functin $f(x) = ax^2 + bx + c$ satisfies $|f(x)| \leq 1$ for all $x \in [0,1]$. Find the greatest possible value of $|a| + |b| + |c|$.

2. Does there exist a positive integer $n > 1$ such that n is a power of 2 and one of the numbers obtained by permuting its (decimal) digits is a power of 3?

3. Prove that for any $n \in \mathbb{N}$ the number $1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2n+1}$ is not an integer.

4. Let $ABCD$ be a convex quadrilateral and let N on side AD and M on side BC be points such that $AN/ND = BM/MC$. The lines AM and BN intersect at P, while the lines CN and DM intersect at Q. Prove that if $S_{ABP} + S_{CDQ} = S_{MPNQ}$, then either $AD \parallel BC$ or N is the midpoint of AD.

Second Day

5. Integers $a_1, a_2, \ldots, a_9$ satisfy the relations $a_{k+1} = a_k^3 + a_k^2 + a_k + 2$ for $k = 1, 2, \ldots, 8$. Prove that among these numbers there exist three with a common divisor greater than 1.

6. The coefficients of the equation $ax^2 + bx + c = 0$, where $a \neq 0$, satisfy the inequality $(a + b + c)(4a - 2b + c) < 0$. Prove that this equation has two distinct real solutions.

7. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{2a+b} + \frac{b}{2b+c} + \frac{c}{2c+a} \leq 1.$$

8. The circles C_1 and C_2 with centers O_1 and O_2 respectively are externally tangent. Their common tangent not intersecting the segment O_1O_2 touches C_1 at A and C_2 at B. Let C be the reflection of A in O_1O_2 and P be the intersection of AC and O_1O_2. Line BP meets C_2 again at L. Prove that line CL is tangent to the circle C_2.

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Grade 10

First Day

1. We are given three nuggets of weights 1kg, 2kg and 3kg, containing different percentages of gold, and need to cut each nugget into two parts so that the obtained parts can be alloyed into two ingots of weights 1kg and 5kg containing the same proportion of gold. How can we do that?

2. Let $n \geq 3$ distinct non-collinear points be given on a plane. Show that there is a closed simple polygonal line passing through each point.

3. The incircle of a triangle ABC is centered at I and tangent to sides AB, BC, CA at C_1, A_1, B_1, respectively. If B_2 is the midpoint of the side BC, prove that the lines B_1I, A_1C_1, and BB_2 have a common point.

4. The numbers $1, 2, \dots, 2n+1$ are written in the cells of a $(2n+1) \times (2n+1)$ board so that the numbers in each row or column are distinct. Suppose that the arrangement of the numbers is symmetric with respect to the main diagonal. Show that all numbers on the main diagonal are distinct.

Second Day

5. Find all triples of prime numbers of the form $(p, 2p+1, 4p+1)$.

6. Let a, b, c be real numbers with $a \geq b \geq c > 1$. Prove the inequality

$$\log_c \log_c b + \log_b \log_b a + \log_a \log_a c \geq 0.$$

7. There are 16 persons in a company, each of which likes exactly 8 other persons. Show that there exist two persons in this company who like each other.

8. In a triangle ADB_1, the angle at A is not right. Squares $ABCD$ and $AB_1C_1D_1$ with centers O_1 and O_2, respectively, are constructed in the exterior of the triangle. Prove that the circumcircles of triangles BAB_1, DAD_1 and O_1AO_2 have a common point other than A.

Grade 11

First Day

1. The sequence (a_n) is defined by $a_1 \in (0,1)$ and $a_{n+1} = a_n(1-a_n)$ for $n \geq 1$. Prove that $\lim\limits_{n\to\infty} na_n = 1$.

2. For every nonnegative integer n and every real number x prove the inequality
$$|\cos x| + |\cos 2x| + |\cos 4x| + \cdots + |\cos 2^n x| \geq \frac{n}{2\sqrt{2}}.$$

3. Let $\mathscr{P}$ be a polyhedron whose all edges are congruent and tangent to a sphere. Suppose that one of the faces of $\mathscr{P}$ has an odd number of sides. Prove that all vertices of $\mathscr{P}$ lie on a single sphere.

4. At least two of the nonnegative real numbers $a_1, a_2, \dots, a_n$ are nonzero. Decide whether a or b is larger, where
$$a = \sqrt[2002]{a_1^{2002} + \cdots + a_n^{2002}} \quad \text{or} \quad b = \sqrt[2003]{a_1^{2003} + \cdots + a_n^{2003}}.$$

Second Day

5. Solve in $\mathbb{R}$ the equation $\sqrt{1-x} = 2x^2 - 1 + 2x\sqrt{1-x^2}$.

6. Can a square of side 1024 be partitioned into 31 squares? Can a square of side 1023 be partitioned into 30 squares, one of which has a side length not exceeding 1?

7. Let a and b be distinct positive numbers. Prove the inequality
$$\sqrt{ab} < \frac{a-b}{\ln a - \ln b} < \frac{a+b}{2}.$$

8. The circumradius of a tetrahedron $ABCD$ is R, and the lengths of the segments connecting the vertices A, B, C, D with the centroids of the opposite faces are equal to m_a, m_b, m_c, and m_d, respectively. Prove that
$$m_a + m_b + m_c + m_d \leq \frac{16}{3}R.$$
When does equality hold?

Grade 12

First Day

1. Consider the function $f:[1,\infty)\to\mathbb{R}$, $f(x)=\int_1^x \frac{dt}{t+2002t^{2002}}$. Prove that for any n real numbers $x_1,x_2,\dots,x_n\geq 1$,
$$\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n}\leq \ln\frac{x_1+x_2+\cdots+x_n}{n}.$$

2. Let (x_1,y_1), (x_2,y_2), (x_3,y_3) be three distinct collinear points of the set $\Gamma=\{(x,y)\in\mathbb{R}^2 \mid x^3=y^2\neq 0\}$. Prove that $\frac{x_1}{y_1}+\frac{x_2}{y_2}+\frac{x_3}{y_3}=0$.

3. Solve the equation $x^5-20x^3+80x-80=0$ in the complex numbers.

4. A sequence of positive integers is written on the blackboard as follows: The first term is 4, and every further term is obtained by multiplying its precedent by 3 and adding 4. How many terms divisible by 2002 are there among the first 2002 terms?

Second Day

5. Real numbers a,b,c satisfy $0\leq a\leq b\leq c\leq 3$. Prove the inequality
$$(a-b)(a^2-9)+(a-c)(b^2-9)+(b-c)(c^2-9)\leq 36.$$

6. Let A,B,C be distinct points on a line, and Γ be the circle with center A and radius r. Points M and N on circle Γ are diametrically opposite. Find the locus of the intersection point of the lines BM and CN.

7. If (x,y) is a point on the ellipse $\mathscr{E}:x^2+9y^2=18$, find the smallest possible value of the expression $x^2+3xy+9y^2+x+3y$.

8. For $n\in\mathbb{N}$, define $a_n=\sin^3\frac{\pi}{3}+3\sin^3\frac{\pi}{3^2}+\cdots+3^{n-1}\sin^3\frac{\pi}{3^n}$. Prove that the sequence (a_n) converges when $n\to\infty$ and find its limit.