45-th Moldova Mathematical Olympiad 2001

Final Round – Chişinău, March 11–12

Grade 7

First Day

1. Prove that $y\sqrt{3-2x}+x\sqrt{3-2y} \le x^2+y^2$ for any numbers $x, y \in [1, \frac{3}{2}]$. When does equality occur?

2. Let $S(n)$ denote the sum of digits of a natural number n. Find all n for which $n+S(n) = 2004$.

3. A line d_i $(i = 1, 2, 3)$ intersects two opposite sides of a square $ABCD$ at points M_i and N_i. Prove that if $M_1N_1 = M_2N_2 = M_3N_3$, then two of the lines d_i are either parallel or perpendicular.

4. Find all permutations of the numbers $1, 2, \dots, 9$ in which no two adjacent numbers have a sum divisible by 7 or 13.

Second Day

5. Let a, b, c, d be real numbers. Prove that the set $M = \{ax^3 + bx^2 + cx + d \mid x \in \mathbb{R}\}$ contains no irrational numbers if and only if $a = b = c = 0$ and d is rational.

6. Two sides of a quadrilateral $ABCD$ are parallel. Let M and N be the midpoints of BC and CD respectively, and P be the intersection point of AN and DM. Prove that if $AP = 4PN$, then $ABCD$ is a parallelogram.

7. Let n be a positive integer. We denote by S the sum of elements of the set $M = \{x \in \mathbb{N} \mid (n-1)^2 \le x < (n+1)^2\}$.
   (a) Show that S is divisible by 6.
   (b) Find all $n \in \mathbb{N}$ for which $S + (1-n)(1+n) = 2001$.

8. Prove that every positive integer k can be written as $k = \dfrac{mn+1}{m+n}$, where m, n are positive integers.

Grade 8

First Day

1. Prove that $\dfrac{1}{2002} < \dfrac{1}{2} \cdot \dfrac{3}{4} \cdot \dfrac{5}{6} \cdots \dfrac{2001}{2002} < \dfrac{1}{44}$.

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2. If $n \in \mathbb{N}$ and $a_1, a_2, \dots, a_n$ are arbitrary numbers in the interval $[0,1]$, find the maximum possible value of the smallest among the numbers $a_1 - a_1a_2, a_2 - a_2a_3, \dots, a_n - a_na_1$.

3. In a triangle ABC, the line symmetric to the median through A with respect to the bisector of the angle at A intersects BC at M. Points P on AB and Q on AC are chosen such that $MP \parallel AC$ and $MQ \parallel AB$. Prove that the circumcircle of the triangle MPQ is tangent to the line BC.

4. Find all integers that can be written as $\dfrac{(a+b)(b+c)(c+a)}{abc}$, where a,b,c are pairwise coprime positive integers.

Second Day

5. Consider all quadratic trinomials $x^2 + px + q$ with $p,q \in \{1,\dots,2001\}$. Which of them are more numbered: those having integer roots, or those having no real roots?

6. Find the intersection of all sets of consecutive positive integers having at least four elements and the sum of elements equal to 2001.

7. The incircle of a triangle ABC is centered at O and touches AC, AB and BC at points K, L, M, respectively. The median BB_1 of $\triangle ABC$ intersects MN at D. Prove that the points O, D, K are collinear.

8. Let S be the set of positive integers x for which there exist positive integers y and m such that $y^2 - 2^m = x^2$.

   (a) Find all the elements of S.

   (b) Find all x such that both x and $x+1$ are in S.

Grade 9

First Day

1. Real numbers $b > a > 0$ are given. Find the number r in $[a,b]$ which minimizes the value of $\max\left\{\left|\frac{r-x}{x}\right| \mid a \le x \le b\right\}$.

2. Prove that the sum of two consecutive prime numbers is never a product of two prime numbers.

3. During a fight each of the 38 cocks has torn out exactly one feather of another cock, and each cock has lost a feather. It turned out that among any three cocks there is one who hasn't torn out a feather from any of the other two cocks. Show that it is possible to kill 6 cocks and place the rest into two henhouses in such a way that no two cocks, one of which has torn out a feather from the other one, stay in the same henhouse.

4. In a triangle ABC the altitude AD is drawn. Points M on side AC and N on side AB are taken so that $\angle MDA = \angle NDA$. Prove that the lines AD, BM and CN are concurrent.

Second Day

5. Show that there are nine distinct nonzero integers such that their sum is a perfect square and the sum of any eight of them is a perfect cube.

6. Prove that for any integer $n > 1$ there are distinct integers a, b, c between n^2 and $(n+1)^2$ such that c divides $a^2 + b^2$.

7. A line is drawn through a vertex of a triangle and cuts two of its middle lines (i.e. lines connecting the midpoints of two sides) in the same ratio. Determine this ratio.

8. Suppose that a, b, c are real numbers such that $|ax^2 + bx + c| \leq 1$ for $-1 \leq x \leq 1$. Prove that $|cx^2 + bx + a| \leq 2$ for $-1 \leq x \leq 1$.

Grade 10

First Day

1. Find all real solutions of the equation

$$x^2 + y^2 + z^2 + t^2 = xy + yz + zt + t - \frac{2}{5}.$$

2. Prove that there are no 2003 odd positive integers whose product equals their sum. Is the previous proposition true for 2001 odd positive integers?

3. During a fight each of the 2001 cocks has torn out exactly one feather of another cock, and each cock has lost a feather. It turned out that among any three cocks there is one who hasn't torn out a feather from any of the other two cocks. Find the smallest k with the following property: It is always possible to kill k cocks and place the rest into two henhouses in such a way that no two cocks, one of which has torn out a feather from the other one, stay in the same henhouse.

4. In a triangle ABC, the angle bisector at A intersects BC at D. The tangents at D to the circumcircles of the triangles ABD and ACD meet AC and AB at N and M, respectively. Prove that the quadrilateral $AMDN$ is inscribed in a circle tangent to BC.

Second Day

5. Let a, b, c be real numbers such that $|ax^2 + bx + c| \leq 1$ for $-1 \leq x \leq 1$. Prove that $|cx^2 + bx + a| \leq 2$ for $-1 \leq x \leq 1$.

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6. Set $a_n = \dfrac{2n}{n^4+3n^2+4}$, $n \in \mathbb{N}$. Prove that $\dfrac{1}{4} \le a_1 + a_2 + \cdots + a_n \le \dfrac{1}{2}$ for all n.

7. Let $ABCD$ and $AB'C'D'$ be equally oriented squares. Prove that the lines BB_1, CC_1, DD_1 are concurrent.

8. A box $3 \times 5 \times 7$ is divided into unit cube cells. In each of the cells there is a cockchafer. At a signal every cockchafer moves through a face of its cell to a neighboring cell.

   (a) What is the minimum number of empty cells after the signal?

   (b) The same question, assuming that the cockchafers move to diagonally adjacent cells (sharing exactly one vertex).

Grade 11

First Day

1. Consider the set $M = \{1, 2, \dots, n\}$, $n \in \mathbb{N}$. Find the smallest positive integer k with the following property: In every k-element subset S of M there exist two elements, one of which divides the other one.

2. Let $m \ge 2$ be an integer. The sequence $(a_n)_{n \in \mathbb{N}}$ is defined by $a_0 = 0$ and $a_n = [n/m] + a_{[n/m]}$ for all n. Determine $\lim\limits_{n \to \infty} \dfrac{a_n}{n}$.

3. For an arbitrary point D on side BC of an acute-angled triangle ABC, let O_1 and O_2 be the circumcenters of the triangles ABD and ACD, and O be the circumcenter of the triangle AO_1O_2. Find the locus of O when D runs over side BC.

4. Let $P(x) = x^n + a_1x^{n-1} + \cdots + a_n$ $(n \ge 2)$ be a polynomial with real coefficients having n real roots $b_1, \dots, b_n$. Prove that for $x_0 \ge \max\{b_1, \dots, b_n\}$,

$$P(x_0+1)\left(\frac{1}{x_0-b_1} + \cdots + \frac{1}{x_0-b_n}\right) \ge 2n^2.$$

Second Day

5. Prove that the sum of the numbers $1, 2, \dots, n$ divides their product if and only if $n+1$ is a composite number.

6. For a positive integer n, denote $A_n = \{(x,y) \in \mathbb{Z}^2 \mid x^2 + xy + y^2 = n\}$.

   (a) Prove that the set A_n is always finite.

   (b) Prove that the number of elements of A_n is divisible by 6 for all n.

   (c) For which n is the number of elements of A_n divisible by 12?

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7. Set $a_n = \dfrac{2n}{n^4+3n^2+4}$, $n \in \mathbb{N}$. Prove that the sequence $S_n = a_1 + a_2 + \cdots + a_n$ is bounded and find its limit.

8. If $a_1, a_2, \ldots, a_n$ are positive real numbers, prove the inequality

$$\frac{1}{\dfrac{1}{1+a_1} + \dfrac{1}{1+a_2} + \cdots + \dfrac{1}{1+a_n}} - \frac{1}{\dfrac{1}{a_1} + \dfrac{1}{a_2} + \cdots + \dfrac{1}{a_n}} \geq \frac{1}{n}.$$

Grade 12

First Day

1. The sequence of functions $f_n : [0,1] \to \mathbb{R}$ $(n \geq 2)$ is given by $f_n = 1 + x^{n^2-1} + x^{n^2+2n}$. Let S_n denote the area of the the figure bounded by the graph of the function f_n and the lines $x = 0$, $x = 1$ and $y = 0$. Compute

$$\lim_{n\to\infty} \left(\frac{\sqrt{S_1} + \sqrt{S_2} + \cdots + \sqrt{S_n}}{n} \right)^n.$$

2. A regular n-gon is inscribed in a unit circle. Compute the product from a fixed vertex to all the other vertices.

3. Find all polynomials $P(x)$ with real coefficieints such that $P(x^2) = P(x)P(x-1)$ for all $x \in \mathbb{R}$.

4. In a triangle ABC, $BC = a$, $AC = b$, $\angle B = \beta$ and $\angle C = \gamma$. Prove that the bisector of the angle at A is equal to the altitude from B if and only if $b = a \cos \dfrac{\beta - \gamma}{2}$.

Second Day

5. For each integer $n \geq 2$ prove the inequality

$$\log_2 3 + \log_3 4 + \cdots + \log_n (n+1) < n + \ln n - 0.9.$$

6. Prove that if a positive integer n divides the five-digit numbers $\overline{a_1a_2a_3a_4a_5}$, $\overline{b_1b_2b_3b_4b_5}$, $\overline{c_1c_2c_3c_4c_5}$, $\overline{d_1d_2d_3d_4d_5}$, $\overline{e_1e_2e_3e_4e_5}$, then it also divides the determinant

$$D = \begin{vmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ b_1 & b_2 & b_3 & b_4 & b_5 \\ c_1 & c_2 & c_3 & c_4 & c_5 \\ d_1 & d_2 & d_3 & d_4 & d_5 \\ e_1 & e_2 & e_3 & e_4 & e_5 \end{vmatrix}.$$

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7. Let $f : [0,1] \to \mathbb{R}$ be a continuously differentiable function such that $f(x_0) = 0$ for some $x_0 \in [0,1]$. Prove that

$$\int_0^1 f(x)^2\,dx \leq 4\int_0^1 f'(x)^2\,dx.$$

8. Let P be the midpoint of the arc AC of a circle, and B be a point on the arc AP. Let M and N be the projections of P onto the segments AC and BC respectively. Prove that if D is the intersection of the bisector of $\angle ABC$ and the segment AC, then every diagonal of the quadrilateral $BDMN$ bisects the area of the triangle ABC.

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