14-th Macedonian Mathematical Olympiad 2007

April 14

1. Let a,b,c be positive real numbers. Prove that

$$1+\frac{3}{ab+bc+ca}\geq\frac{6}{a+b+c}.$$

2. In a trapezoid $ABCD$ with a base AD, point L is the orthogonal projection of C on AB, and K is the point on BC such that AK is perpendicular to AD. Let O be the circumcenter of triangle ACD. Suppose that the lines AK, CL and DO have a common point. Prove that $ABCD$ is a parallelogram.

3. Natural numbers a,b and c are pairwise distinct and satisfy

$$a\mid b+c+bc,\quad b\mid c+a+ca,\quad c\mid a+b+ab.$$

   Prove that at least one of the numbers a,b,c is not prime.

4. Find all functions $f:\mathbb{R}\to\mathbb{R}$ that satisfy

$$f(x^3+y^3)=x^2f(x)+yf(y^2)\quad\text{for all } x,y\in\mathbb{R}.$$

5. Let n be a natural number divisible by 4. Determine the number of bijections f on the set $\{1,2,\dots,n\}$ such that $f(j)+f^{-1}(j)=n+1$ for $j=1,\dots,n$.

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