11-th Korean Mathematical Olympiad 1997/98

Final Round

First Day – April 18, 1998

1. Find all pairwise coprime positive integers l, m, n such that $(l+m+n)\left(\frac{1}{l}+\frac{1}{m}+\frac{1}{n}\right)$ is an integer.

2. Let D, E, F be points on the sides BC, CA, AB respectively of a triangle ABC. Lines AD, BE, CF intersect the circumcircle of ABC again at P, Q, R, respectively. Show that
$$\frac{AD}{PD}+\frac{BE}{QE}+\frac{CF}{RF} \geq 9,$$
and find the cases of equality.

3. For $n \in \mathbb{N}$, let $\varphi(n)$ denote the Euler function of n and let $\psi(n)$ denote the number of prime divisors of n. Show that if $\varphi(n) \mid n-1$ and $\psi(n) \leq 3$, then n is prime.

Second Day – April 19, 1998

4. Let a, b, c be positive real numbers satisfying $a+b+c = abc$. Prove that
$$\frac{1}{\sqrt{1+a^2}}+\frac{1}{\sqrt{1+b^2}}+\frac{1}{\sqrt{1+c^2}} \leq \frac{3}{2},$$
and find when equality occurs.

5. Let I be the incenter of triangle ABC, O_1 a circle through B tangent to CI, and O_2 a circle through C tangent to BI. Prove that O_1, O_2 and the circumcircle of ABC have a common point.

6. Let F_n be the set of bijective functions from $\{1, 2, \dots, n\}$ to itself such that

   (a) $f(k) \leq k+1$ for all k;

   (b) $f(k) \neq k$ for $2 \leq k \leq n$.

   Find the probability that $f(1) \neq 1$ for f randomly chosen from F_n.

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