10-th Korean Mathematical Olympiad 1996/97

Final Round

First Day – April 19, 1997

1. A word is a sequence of 0 and 1 of length 8. Let \(x\) and \(y\) be two words differing in exactly three places. Prove that the number of words differing from each of \(x\) and \(y\) in at least five places is 188.

2. The incircle of a triangle \(A_1A_2A_3\) is centered at \(O\) and meets the segment \(OA_j\) at \(B_j\), \(j = 1,2,3\). A circle with center \(B_j\) is tangent to the two sides of the triangle having \(A_j\) as an endpoint and intersects the segment \(OB_j\) at \(C_j\). Prove that

\[
OC_1 + OC_2 + OC_3 \leq \frac{1}{4\sqrt{3}} A_1A_2 + A_2A_3 + A_3A_1
\]

and find the conditions for equality.

3. Find all pairs of functions \(f, g : \mathbb{R} \to \mathbb{R}\) such that

(i) if \(x < y\), then \(f(x) < f(y)\);
(ii) \(f(xy) = g(y)f(x) + f(y)\) for all \(x, y \in \mathbb{R}\).

Second Day – April 20, 1997

4. Given a positive integer \(n\), find the number of \(n\)-digit natural numbers consisting of digits 1, 2, 3 in which any two adjacent digits are either distinct or both equal to 3.

5. For positive numbers \(a_1, a_2, \ldots, a_n\), we define

\[
A = \frac{a_1 + \cdots + a_n}{n}, \quad G = \sqrt[n]{a_1 \cdots a_n}, \quad H = \frac{n}{a_1^{-1} + \cdots + a_n^{-1}}.
\]

Prove that

\[
\begin{cases}
\frac{A}{H} \leq -1 + \left(\frac{A}{G}\right)^n & \text{for } n \text{ even}; \\
\frac{A}{H} \leq \frac{n-2}{n} + \frac{2(n-1)}{n} \left(\frac{A}{G}\right)^n & \text{for } n \text{ odd}.
\end{cases}
\]

6. Let \(p_1, p_2, \ldots, p_r\) be distinct primes, and let \(n_1, n_2, \ldots, n_r\) be arbitrary natural numbers. Prove that the number of pairs of integers \((x, y)\) such that

\[
x^3 + y^3 = p_1^{n_1} p_2^{n_2} \cdots p_r^{n_r}
\]

does not exceed \(2^{r+1}\).