8-th Korean Mathematical Olympiad 1994/95

Final Round

First Day – April 15, 1995

1. Show that for any positive integer m, there exist integers a,b satisfying

$$|a|,|b| \leq m, \quad 0 < a + b\sqrt{2} \leq \frac{1+\sqrt{2}}{m+2}.$$

2. Let A denote the set of nonnegative integers. Find all functions $f : A \to A$ satisfying the following two conditions:

   (i) $2f(m^2+n^2) = f(m)^2 + f(n)^2$ for all $m,n \in A$;

   (ii) $f(m^2) \geq f(n^2)$ for any $m,n \in A$ with $m \geq n$.

3. Let ABC be an equilateral triangle of side 1, D be a point on BC, and r_1, r_2 be the inradii of triangles ABD and ADC. Express $r_1 r_2$ in terms of $p = BD$ and find the maximum of $r_1 r_2$.

Second Day – April 16, 1995

4. Let O and R be the circumcenter and circumradius of a triangle ABC, and let P be any point in the plane of the triangle. The perpendiculars PA_1, PB_1, PC_1 are dropped from P to BC, CA, AB. Express $S_{A_1B_1C_1}/S_{ABC}$ in terms of R and $d = OP$, where S_{XYZ} is the area of $\triangle XYZ$.

5. Let a,b be integers and p be a prime number such that:

   (i) p is the greatest common divisor of a and b;

   (ii) p^2 divides a.

   Prove that the polynomial $x^{n+2} + ax^{n+1} + bx^n + a + b$ cannot be decomposed into the product of two polynomials with integer coefficients and degree greater than 1.

6. Let m,n be positive integers with $1 \leq n < m$. A box is locked with several padlocks which must all be opened to open the box, and which all have different keys. The keys are distributed among m people. Suppose that among these people, no n can open the box, but any $n+1$ can open it. Find the smallest possible number l of locks and then the total number of keys for which this is possible.

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