7-th Korean Mathematical Olympiad 1994

Final Round

First Day – April 16, 1994.

1. Let S be the set of all nonnegative integers. Determine all functions $f,g,h : S \to S$ satisfying the following two conditions:

   (i) $f(m+n) = g(m) + h(n) + 2mn$ for any $m,n \in S$;

   (ii) $g(1) = h(1) = 1$.

2. Let α, β, γ be the angles of a triangle. Prove that
$$\csc^2 \frac{\alpha}{2} + \csc^2 \frac{\beta}{2} + \csc^2 \frac{\gamma}{2} \geq 12$$
and find the conditions for equality.

3. In a triangle ABC, I and O are the incenter and circumcenter respectively, A', B', C' the excenters, and O' the circumcenter of $\triangle A'B'C'$. If R and R' are the circumradii of triangles ABC and $A'B'C'$, respectively, prove that
$$\text{(i) } R' = 2R \quad \text{and} \quad \text{(ii) } IO' = 2IO.$$

Second Day – April 17, 1994.

4. Consider the equation $y^2 - k = x^3$, where k is an integer.

   Prove that the equation cannot have five integer solutions of the form
$$(x_1, y_1), (x_2, y_1 - 1), (x_3, y_1 - 2), (x_4, y_1 - 3), (x_5, y_1 - 4).$$
   Also show that if it has the first four of these pairs as integer solutions, then $k \equiv 17 \pmod{63}$.

5. Given a set $S \subset \mathbb{N}$ and a positive integer n, let $S \oplus \{n\} = \{s + n \mid s \in S\}$. The sequence S_k of sets is defined inductively as follows:
$$S_1 = \{1\}, \quad S_k = (S_{k-1} \oplus \{k\}) \cup \{2k-1\} \text{ for } k = 2,3,4,\ldots$$
   (a) Determine $\mathbb{N} \setminus \bigcup_{k=1}^{\infty} S_k$.

   (b) Find all n for which $1994 \in S_n$.

6. Let α, β, γ be the angles of $\triangle ABC$.

   (a) Show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 - 2\cos\alpha\cos\beta\cos\gamma$.

   (b) Given that $\cos\alpha : \cos\beta : \cos\gamma = 39 : 33 : 25$, find $\sin\alpha : \sin\beta : \sin\gamma$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com