20-th Korean Mathematical Olympiad 2007

Final Round

First Day – March 24, 2007

1. Let O be the circumcenter of an acute triangle ABC and let k be the circle with center O' that is tangent to O at A and tangent to side BC at D. Circle k meets AB and AC again at E and F respectively. The lines OO' and EO' meet k again at A' and G. Lines BO and $A'G$ intersect at H. Prove that $DF^2 = AF \cdot GH$.

2. How many ways are there to write either 0 or 1 in each cell of a 4×4 board so that the product of numbers in any two cells sharing an edge is always 0?

3. Find all triples (x, y, z) of positive integers satisfying $1 + 4^x + 4^y = z^2$.

Second Day – March 25, 2007

4. Find all pairs (p, q) of primes such that $p^p + q^q + 1$ is divisible by pq.

5. For the vertex A of a triangle ABC, let l_a be the distance between the projections on AB and AC of the intersection of the angle bisector of $\angle A$ with side BC. Define l_b and l_c analogously. If l is the perimeter of triangle ABC, prove that
$$\frac{l_a l_b l_c}{l^3} \leq \frac{1}{64}.$$

6. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying $kf(n) \leq f(kn) \leq kf(n) + k - 1$ for all $k, n \in \mathbb{N}$.

   (a) Prove that $f(a) + f(b) \leq f(a+b) \leq f(a) + f(b) + 1$ for all $a, b \in \mathbb{N}$.

   (b) If f satisfies $f(2007n) \leq 2007f(n) + 2005$ for every $n \in \mathbb{N}$, show that there exists $c \in \mathbb{N}$ such that $f(2007c) = 2007f(c)$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com