19-th Korean Mathematical Olympiad 2006

Final Round

First Day – March 25, 2006

1. In a triangle ABC with $AB \neq AC$, the incircle touches the sides BC, CA, AB at D, E, F, respectively. Line AD meets the incircle again at P. The line EF and the line through P perpendicular to AD meet at Q. Line AQ intersects DE at X and DF at Y. Prove that A is the midpoint of XY.

2. For a positive integer a, let S_a be the set of primes p for which there exists an odd integer b such that p divides $\left(2^{2^a}\right)^b - 1$. Prove that for every a there exist infinitely many primes that are not contained in S_a.

3. Three schools A, B and C, each with five players denoted a_i, b_i, c_i respectively, take part in a chess tournament. The tournament is held following the rules:

   (i) Players from each school have matches in order with respect to indices, and defeated players are eliminated; the first match is between a_1 and b_1.

   (ii) If $y_j \in Y$ defeats $x_i \in X$, his next opponent should be from the remaining school if not all of its players are eliminated; otherwise his next oponent is x_{i+1}. The tournament is over when two schools are completely eliminated.

   (iii) When x_i wins a match, its school wins 10^{i-1} points.

   At the end of the tournament, schools A, B, C scored P_A, P_B, P_C respectively. Find the remainder of the number of possible triples (P_A, P_B, P_C) upon division by 8.

Second Day – March 26, 2006

4. Given three distinct real numbers a_1, a_2, a_3, define

$$b_j = \left(1 + \frac{a_j a_i}{a_j - a_i}\right)\left(1 + \frac{a_j a_k}{a_j - a_k}\right), \quad \text{where } \{i, j, k\} = \{1, 2, 3\}.$$

   Prove that $1 + |a_1 b_1 + a_2 b_2 + a_3 b_3| \leq (1 + |a_1|)(1 + |a_2|)(1 + |a_3|)$ and find the cases of equality.

5. In a convex hexagon $ABCDEF$ triangles ABC, CDE, EFA are similar. Find conditions on these triangles under which triangle ACE is equilateral if and only if so is BDF.

6. A positive integer N is said to be *n-good* if

   (i) N has at least n distinct prime divisors, and

   (ii) there exist distinct positive divisors $1, x_2, \ldots, x_n$ whose sum is N.

   Show that there exists an n-good number for each $n \geq 6$.

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