19-th Korean Mathematical Olympiad 2006

Final Round

First Day – March 25, 2006

1. In a triangle $ABC$ with $AB \neq AC$, the incircle touches the sides $BC, CA, AB$ at $D, E, F$, respectively. Line $AD$ meets the incircle again at $P$. The line $EF$ and the line through $P$ perpendicular to $AD$ meet at $Q$. Line $AQ$ intersects $DE$ at $X$ and $DF$ at $Y$. Prove that $A$ is the midpoint of $XY$.

2. For a positive integer $a$, let $S_a$ be the set of primes $p$ for which there exists an odd integer $b$ such that $p$ divides $(2^a)^b - 1$. Prove that for every $a$ there exist infinitely many primes that are not contained in $S_a$.

3. Three schools $A, B$ and $C$, each with five players denoted $a_1, b_1, c_1$ respectively, take part in a chess tournament. The tournament is held following the rules:

   (i) Players from each school have matches in order with respect to indices, and defeated players are eliminated; the first match is between $a_1$ and $b_1$.

   (ii) If $y_j \in Y$ defeats $x_i \in X$, his next opponent should be from the remaining school if not all of its players are eliminated; otherwise his next opponent is $x_{i+1}$. The tournament is over when two schools are completely eliminated.

   (iii) When $x_i$ wins a match, its school wins $10^{i-1}$ points.

At the end of the tournament, schools $A, B, C$ scored $P_A, P_B, P_C$ respectively. Find the remainder of the number of possible triples $(P_A, P_B, P_C)$ upon division by 8.

Second Day – March 26, 2006

4. Given three distinct real numbers $a_1, a_2, a_3$, define

   \[ b_j = \left( 1 + \frac{a_j a_i}{a_j - a_i} \right) \frac{1 + a_j a_k}{a_j - a_k}, \text{ where } \{i, j, k\} = \{1, 2, 3\}. \]

Prove that $1 + |a_1 b_1 + a_2 b_2 + a_3 b_3| \leq (1 + |a_1|)(1 + |a_2|)(1 + |a_3|)$ and find the cases of equality.

5. In a convex hexagon $ABCDEF$ triangles $ABC$, $CDE$, $EFA$ are similar. Find conditions on these triangles under which triangle $ACE$ is equilateral if and only if so is $BDF$.

6. A positive integer $N$ is said to be $n$-good if

   (i) $N$ has at least $n$ distinct prime divisors,

   (ii) there exist distinct positive divisors $1, x_2, \ldots, x_n$ whose sum is $N$.

Show that there exists an $n$-good number for each $n \geq 6$. 

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