18-th Korean Mathematical Olympiad 2005

Final Round

First Day – April 9, 2005

1. Find all positive integers n that can be uniquely expressed as a sum of at most five nonzero perfect squares. (The order of the summands is irrelevant.)

2. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive real numbers and let α_n be the arithmetic mean of $a_1, \dots, a_n$. Prove that for all positive integers N,

$$\sum_{n=1}^{N} \alpha_n^2 \leq 4 \sum_{n=1}^{N} a_n^2.$$

3. In a trapezoid $ABCD$ with $AD \parallel BC$, O_1, O_2, O_3, O_4 denote the circles with diameters AB, BC, CD, DA, respectively. Show that there exists a circle with center inside the trapezoid which is tangent to all the four circles $O_1, \dots, O_4$ if and only if $ABCD$ is a parallelogram.

Second Day – April 10, 2005

4. Let O be the circumcircle of a triangle ABC with $\angle A = 90^\circ$ and $\angle B > \angle C$. The line l_A tangent to O at A meets BC at S, the line l_B tangent to O at B meets AC at D, and the lines DS and AB meet at E. The line CE intersects l_A at T. Let P be the foot of the perpendicular from E to l_A, let CP intersect O again at Q, and let QT intersect O again at R. If BR and l_A meet at U, prove that $\dfrac{SU \cdot SP}{TU \cdot TP} = \dfrac{SA^2}{TA^2}$.

5. Find all positive integers m and n such that both $3^m + 1$ and $3^n + 1$ are divisible by mn.

6. A set P consists of 2005 distinct prime numbers. Let A be the set of all possible products of 1002 elements of P, and B be the set of all products of 1003 elements of P. Find a one-to-one correspondance f from A to B with the property that a divides $f(a)$ for all $a \in A$.

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