# 17-th Korean Mathematical Olympiad 2004

## Final Round

*First Day – April 10, 2004*

1. The incircle of an isosceles triangle $ABC$ with $AB = AC$ is centered at $O$ and touches $BC, CA, AB$ at $K, L, M$, respectively. Lines $KM$ and $OL$ meet at $N$, and line $BN$ meets $CA$ at $Q$. Let $P$ be the projection of $A$ onto $BQ$. Suppose that $BP = AP + 2PQ$. Find all possible values of $AB/BC$.

2. Show that the equation $3y^2 = x^4 + x$ has no solutions in positive integers.

3. Some 2004 computers are to be connected by cables into a network. A set $S$ of several computers is called *independent* if no two computers in $S$ are connected by a cable. It is required that no independent set consists of more than 50 computers. Suppose that such a network is built using the least possible number of cables.

   (a) Let $c(L)$ denote the number of cables at a computer $L$. Show that for any two computers $A$ and $B$ we have $|c(A) - c(B)| \leq 1$, and $c(A) = c(B)$ if $A$ and $B$ are connected by a cable.

   (b) Find the number of cables used.

*Second Day – April 11, 2004*

4. We are given $n$ distinct points on a circumference. How many ways are there to choose $k$ of these $n$ points such that between any two of them there are at least three of the remaining $n - k$ points (counting clockwise)?

5. Let $R$ be the circumradius and $r$ the inradius of an acute triangle $ABC$ with the largest angle at $A$. Let $M$ be the midpoint of $BC$ and let $X$ be the intersection point of the tangent lines to the circumcircle at $B$ and $C$. Prove that

$$\frac{r}{R} \geq \frac{AM}{AX}.$$

6. For a prime number $p$, consider $f_p(x) = x^{p-1} + \cdots + x + 1$.

   (a) If $m$ is a multiple of $p$, show that there is a prime divisor of $f_p(m)$ that is coprime to $m(m-1)$.

   (b) Prove that there are infinitely many prime numbers of the form $pn + 1$, $n \in \mathbb{N}$.



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