16-th Korean Mathematical Olympiad 2003

Final Round

First Day – April 12, 2003

1. The computers in a computer room are in a network such that every computer is connected by a cable to exactly three others. The computers can exchange data directly or indirectly (via other computers). Let k be the smallest number of computers that need to be removed so that two of the remaining computers can no longer exchange data or there is only one computer left. Let l be the smallest number of cables that need to be removed so that some two of the remaining computers can no longer exchange data. Show that $k = l$.

2. The diagonals of a rhombus $ABCD$ with $\angle A < 90^\circ$ intersect at M. Let $O \neq M$ be a point on the segment MC such that $OB < OC$ and $t = MA/MO$. The circle with center O passing through B and D meets AB again at X and BC again at Y. The diagonal AC intersects DX at P and DY at Q. Express OQ/OP in terms of t.

3. Show that the equation $2x^4 + 2x^2y^2 + y^4 = z^2$ has no solutions in integers with $x \neq 0$.

Second Day – April 13, 2003

4. The incircle of a triangle ABC meets the sides AB, BC, CA at P, Q, R, respectively. Prove that
$$\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \geq 6.$$

5. Let m be a positive integer.
   (a) Prove that if $2^{m+1} + 1$ divides $3^{2^m} + 1$, then $2^{m+1} + 1$ is prime.
   (b) Is the converse true?

6. On a circle are given n distinct points. Let m be a positive integer with $3 \leq m < n/2$ and $(m, n) = 1$. Each point is connected by a segment to the m-th point from it, counting counterclockwise. The obtained segments intersect in I different points inside the circle.
   (a) Find the maximum value of I when the n points take different positions on the circle.
   (b) Prove that $I \geq n$ and show that equality is possible for $m = 3$ and an arbitrary even number $n > 6$.

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