15-th Korean Mathematical Olympiad 2002

Final Round

First Day – April 13, 2002

1. For a prime p of the form $12k+1$ and $\mathbb{Z}_p = \{0,1,2,\dots,p-1\}$, define

$$E_p = \{(a,b) \in \mathbb{Z}_p^2 \mid \ p \nmid 4a^3 + 27b^2\}.$$

We say that elements (a,b) and (a',b') of E_p are *equivalent* if there is a nonzero $c \in \mathbb{Z}_p$ such that $p \mid a' - ac^4$ and $p \mid b' - bc^6$. Find the maximal number of inequivalent elements of E_p.

2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x \in \mathbb{R}$ and $y \in f(\mathbb{R})$

$$f(x-y) = f(x) + xy + f(y).$$

3. The following facts are known on a mathematical contest:

   (i) There were $n \geq 4$ problems;

   (ii) Each problem was solved by exactly four contestants;

   (iii) For any two problems there is exactly one contestant who solved both problems.

   Assuming that there were at least $4n$ contestants, find the minimum value of n for which there always exists a contestant who solved all the problems.

Second Day – April 14, 2002

4. For positive real numbers $a_1,\dots,a_n,b_1,\dots,b_n$ $(n \geq 3)$ with the b_i pairwise distinct, denote $S = a_1 + a_2 + \cdots + a_n$ and $T = b_1 b_2 \cdots b_n$.

   (a) Let $f(x) = (x-b_1)(x-b_2)\cdots(x-b_n)\sum_{j=1}^{n} \frac{a_j}{x-b_j}$. Find the number of distinct real zeroes of the polynomial $f(x)$.

   (b) Prove that $\frac{1}{n-1}\sum_{j=1}^{n}\left(1-\frac{a_j}{S}\right) b_j > \left(\frac{T}{S}\sum_{j=1}^{n}\frac{a_j}{b_j}\right)^{\frac{1}{n-1}}$.

5. In an acute-angled triangle ABC, the altitude from A meets the circumcircle O at D. Let P be a point on O, and let Q be the foot of the perpendicular from P to AB. Prove that if Q is outside O and $2\angle QPB = \angle PBC$, then the points D,P,Q are collinear.

6. Let p_n denote the n-th smallest prime ($p_1 = 2$, $p_2 = 3$, $p_3 = 5$, etc.).

(a) For a given $n \geq 10$, let r be the smallest integer such that $p_r > n-3$, and let $N_s = sp_1p_2\cdots p_{r-1} - 1$ for $s = 1, 2, \dots, p_r$. Show that there exists j, $1 \leq j \leq p_r$, such that none of $p_1, p_2, \dots, p_n$ divides N_j.

(b) Using the result of (a), find all m for which $p_{m+1}^2 < p_1p_2\cdots p_m$.