13-th Korean Mathematical Olympiad 2000

Final Round

First Day – April 15, 2000

1. Prove that for every prime p there exist integers x,y,z,w such that $x^2+y^2+z^2-wp=0$ and $0<w<p$.

2. Determine all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all x,y,
$$f(x^2-y^2)=(x-y)(f(x)+f(y)).$$

3. A rectangle $ABCD$ is inscribed in a circle with center O. The internal bisectors of $\angle ABD$ and $\angle ADB$ meet at P; those of $\angle DAB$, $\angle DBA$ meet at Q; those of $\angle ACD$, $\angle ADC$ meet at R; those of $\angle DAC$, $\angle DCA$ at S. Prove that points P,Q,R,S are concyclic.

Second Day – April 16, 2000

4. Let $p\equiv 1 \pmod 4$ be a prime number. Evaluate $\displaystyle\sum_{k=1}^{p-1}\left(\left[\frac{2k^2}{p}\right]-2\left[\frac{k^2}{p}\right]\right)$.

5. Prove that an $m\times n$ rectangle can be constructed using copies of the L-shape tetramino if and only if $8\mid mn$.

6. Let a,b,c,x,y,z be real numbers such that $a>b>c>0$ and $x>y>z>0$. Prove that
$$\frac{a^2x^2}{(by+cz)(bz+cy)}+\frac{b^2y^2}{(cz+ax)(cx+az)}+\frac{c^2z^2}{(ax+by)(ay+bx)}\geq\frac{3}{4}.$$

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