12-th Korean Mathematical Olympiad 1998/99

First Round – November 1998

1. Consider the polynomial

$$f(x) = (1+x)^{n_1} + (1+x)^{n_2} + \cdots + (1+x)^{n_k},$$

where $n_1, n_2, \dots, n_k$ are positive integers. For each positive integer a, find the minimum of the coefficient at x_a as the k-tuple $(n_1, \dots, n_k)$ varies over all k-tuples with sum n.

2. For all natural numbers m, n, prove that $\dfrac{1}{\sqrt[m]{n+1}} + \dfrac{1}{\sqrt[n]{m+1}} \geq 1.$

3. Equilateral triangles ABM and CDP are drawn outside a convex quadrilateral $ABCD$, and equilateral triangles BCN and DAQ are drawn inside $ABCD$. Suppose that among the points M, N, P, Q, no three are on a line. Prove that $MNPQ$ is a parallelogram.

4. On the coordinate plane, a piece can move 1 right or 1 up per move. Find the number of possible ways for the piece to move from point $(0,0)$ to (n,n), not passing through any of the points $(1,1), (2,2), \dots, (n-1, n-1)$.

5. Consider the matrix $A = (a_{ij})_{i,j=1}^{1999}$ given by $a_{ij} = 1$ if $i \geq j$ and $a_{ij} = 0$ otherwise. Find the number of ways of choosing 1998 ones from the matrix so that no two of them are in the same row or column.

6. Let a, b, c be positive real numbers with $abc \geq 1$. Prove that

$$\frac{1}{a+b^4+c^4} + \frac{1}{a^4+b+c^4} + \frac{1}{a^4+b^4+c} \leq 1.$$

7. If I is the incenter of a triangle ABC, prove that

$$IA^2 + IB^2 + IC^2 \geq \frac{AB^2 + BC^2 + CA^2}{3}.$$

8. Prove that the equation $x^3 + y^3 = 7z^3$ has infinitely many solutions in integers x, y, z with $\gcd(x, y, z) = 1$.

The IMO Compendium Group,
D. Djukić, V. Janković, I. Matić, N. Petrović
www.imomath.com