10-th Korean Mathematical Olympiad 1996/97

First Round – November 10, 1996

1. Show that among any four points in a unit circle, there exist two whose distance is at most $\sqrt{2}$.

2. Suppose $f : \mathbb{N} \to \mathbb{N}$ is a function satisfying:

   (i) $f(n + f(n)) = f(n)$ for all n;

   (ii) For some $n_0 \in \mathbb{N}$, $f(n_0) = 1$.

   Prove that $f(n) = 1$ for all $n \in \mathbb{N}$.

3. Express $\sum_{k=1}^{n}[\sqrt{k}]$ in terms of n and $a = [\sqrt{n}]$.

4. A circle C touches the edges of an angle XOY, and a circle C_1 touches these edges and passes through the center of C. Let A be the second endpoint of the diameter of C_1 containing the center of C, and let B be the second intersection point of this diameter with C. Prove that the circle centered at A passing through B touches the edges of $\angle XOY$.

5. Find all integers x, y, z satisfying $x^2 + y^2 + z^2 = 2xyz$.

6. Find the smallest integer k for which there exist two sequences (a_i) and (b_i), $i = 1, 2, \dots, k$, such that:

   (i) $a_i, b_i \in \{1, 1996, 1996^2, \dots\}$ for all i;

   (ii) $a_i \neq b_i$ for all i;

   (iii) $a_i \leq a_{i+1}$ and $b_i \leq b_{i+1}$ for $i = 1, \dots, k-1$;

   (iv) $\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i$.

7. Let A_n be the set of all real nhumbers of the form

   $$1 + \frac{a_1}{\sqrt{2}} + \frac{a_2}{(\sqrt{2})^2} + \cdots + \frac{a_n}{(\sqrt{2})^n}, \quad a_j \in \{-1, 1\}.$$

   Find the number of elements of A_n, and find the sum of all products of two distinct elements of A_n.

8. In an acute triangle ABC with $AB \neq AC$, let V be the intersection of the angle bisector of A with BC, and let D be the foot of the altitude from A. If the circumcircle of $\triangle AVD$ meets CA again at E and AB at F, show that the lines AD, BE, CF concur.

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