8-th Korean Mathematical Olympiad 1994/95

First Round – November 13, 1994

1. On the plane are given finitely many points, such that the area of any triangle with vertices in the given points is always less than 1. Show that all these points lie inside a triangle of area 4.

2. For a given positive integer m, find all pairs (n,x,y) of positive integers such that m,n are coprime and $(x^2+y^2)^m=(xy)^n$.

3. Let A,B,C be points on a given circle, and P,Q,R be the midpoints of arcs BC,CA,AB (not containing A,B,C), respectively. Lines AP,BQ,CR meet BC,CA,AB at L,M,N, respectively. Prove that

$$\frac{AL}{PL}+\frac{BM}{QM}+\frac{CN}{RN}\geq 9.$$

   When does equality hold?

4. A partition of a positive integer n is a sequence of positive integers $(\lambda_1,\dots,\lambda_k)$ such that $\lambda_1+\cdots+\lambda_k=n$ and $\lambda_1\geq\cdots\geq\lambda_k\geq 1$. Each λ_i is called a summand. Show that, for a positive integer m with $m>\frac{1}{2}m(m+1)$, the number of partitions of n into n distinct summands is equal to the number of partitions of $n-\frac{1}{2}m(m+1)$ into at most m summands.

5. If three points are selected at random on a given circle, find the probability that these three points lie on a semicircle.

6. Show that any positive integer $n>1$ can be expressed as a sum of natural numbers satisfying the following conditions:

   (i) none of them has prime factors other than 2 or 3;

   (ii) none of them divides any other.

   In other words, $n=\sum_{i=1}^{N}2^{\alpha_i}3^{\beta_i}$, where $\alpha_i,\beta_i\in\mathbb{N}_0$ and $(\alpha_i-\alpha_j)(\beta_i-\beta_j)<0$ for $i\neq j$.

7. Find all functions $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ that satisfy

$$\frac{1}{x}f(-x)+f\left(\frac{1}{x}\right)=x \quad \text{for all } x\neq 0.$$

8. Two circles O_1 and O_2 of radii r_1 and r_2 respectively intersect at A and B. For any point P on O_1, lines PA and PB meet O_2 at Q and R, respectively.

   (a) Express $y-QR$ in terms of r_1,r_2, and $\theta=\angle APB$.

   (b) Show that $y=2r_2$ is a necessary and sufficient condition for the circles O_1 and O_2 to be orthogonal.

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