9-th Japanese Mathematical Olympiad 1999

Final Round – February 11

1. One can place a stone at each of the squares of a 1999×1999 board. Find the minimum number of stones that must be placed so that, for any blank square on the board, the total number of stones placed in the corresponding row and column is at least 1999.

2. Let $f(x) = x^3 + 17$. Prove that for every integer $n \geq 2$ there exists a natural number x for which $f(x)$ is divisible by 3^n but not by 3^{n+1}.

3. Suppose $2n+1$ weights $(n \in \mathbb{N})$ satisfy the following condition: If any one weight is excluded, the remaining $2n$ weights can be divided into two groups of n weights each which balance each other. Prove that all the weights are equal.

4. Prove that the polynomial $f(x) = (x^2+1^2)(x^2+2^2)\cdots(x^2+n^2)+1$ cannot be expressed as a product of two polynomials with integer coefficients of degree greater than one.

5. All side lengths of a convex hexagon $ABCDEF$ are 1. Let $M = \max\{|AD|, |DE|, |CF|\}$ and $m = \min\{|AD|, |DE|, |CF|\}$. Find possible ranges of M and m.

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